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Hankel operators induced by radial Bekollé–Bonami weights on Bergman spaces

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Abstract

We study big Hankel operators $H_f^v : A_\omega^p \rightarrow L_v^q$ generated by radial Bekollé–Bonami weights v , when $1 < p \leq q < \infty$. Here the radial weight ω is assumed to satisfy a two-sided doubling condition, and A_ω^p denotes the corresponding weighted Bergman space. A characterization for simultaneous boundedness of H_f^v and $H_{\bar{f}}^v$ is provided in terms of a general weighted mean oscillation. Compared to the case of standard weights that was recently obtained by Pau et al. (Indiana Univ Math J 65(5):1639–1673, 2016), the respective spaces depend on the weights ω and v in an essentially stronger sense. This makes our analysis deviate from the blueprint of this more classical setting. As a consequence of our main result, we also study the case of anti-analytic symbols.

Keywords Hankel operator · Bekollé–Bonami weight · Bergman space · Bergman projection · doubling weight

Mathematics Subject Classification Primary 47B35; Secondary 32A36

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1 Introduction and main results

Let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. A function $\omega : \mathbb{D} \rightarrow [0, \infty)$, integrable over the unit disc \mathbb{D} , is called a weight. It is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. For $0 < p < \infty$ and a weight ω , the Lebesgue space L_ω^p consists of (equivalence classes of) complex-valued measurable functions f in \mathbb{D} such that

$$\|f\|_{L_\omega^p} = \left(\int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) \right)^{\frac{1}{p}} < \infty,$$

where $dA(z) = dx dy / \pi$ denotes the normalized Lebesgue area measure on \mathbb{D} . The weighted Bergman space A_ω^p is the space of analytic functions in L_ω^p . As usual, A_α^p denotes the weighted Bergman space induced by the standard radial weight $(\alpha + 1)(1 - |z|^2)^\alpha$. If ν is a radial weight then A_ν^2 is a closed subspace of L_ν^2 , and the orthogonal projection from L_ν^2 to A_ν^2 is given by

$$P_\nu(f)(z) = \int_{\mathbb{D}} f(\zeta) \overline{B_z^\nu(\zeta)} \nu(\zeta) dA(\zeta), \quad z \in \mathbb{D},$$

where B_z^ν are the reproducing kernels of A_ν^2 ; $f(z) = \langle f, B_z^\nu \rangle_{A_\nu^2}$ for all $z \in \mathbb{D}$ and $f \in A_\nu^2$.

The study of the boundedness of weighted Bergman projections on L^p -spaces is a compelling topic that has attracted a considerable amount of attention during the last decades. A well known result due to Bekollé and Bonami [4,5] describes the weights ω such that the Bergman projection P_η , induced by the standard weight $(\eta + 1)(1 - |z|^2)^\eta$, is bounded on L_ω^q for $1 < q < \infty$. We denote this class of weights by $B_q(\eta)$, and write $B_q = \cup_{\eta > -1} B_q(\eta)$ for short. In the case of a standard weight, the Bergman reproducing kernels are given by the neat formula $(1 - \bar{z}\zeta)^{-(2+\eta)}$. However, for a general radial weight ν the Bergman reproducing kernels B_z^ν may have zeros [18] and such explicit formulas for the kernels do not necessarily exist. This is one of the main obstacles in dealing with P_ν [9,16]. Nonetheless, we shall prove in Proposition 6 below that if $\nu \in B_q$ is radial, then $P_\nu : L_\nu^q \rightarrow L_\nu^q$ is bounded for each $1 < q < \infty$. The proof of this relies on accurate estimates for the integral means of B_z^ν recently obtained in [16, Theorem 1], and the result itself plays an important role in the proof of the main discovery of this paper.

All the above makes the class of radial weights in B_q an appropriate framework for the study of the big Hankel operator

$$H_f^\nu(g)(z) = (I - P_\nu)(fg)(z), \quad f \in L_\nu^1, \quad z \in \mathbb{D},$$

on weighted Bergman spaces. For an analytic function f , the Hankel operator H_f^β , induced by a standard projection, has been widely studied on Bergman spaces since the pioneering work of Axler [3], which was later extended in [1]. In the case of a rapidly decreasing weight ν and $f \in \mathcal{H}(\mathbb{D})$, Galanopoulos and Pau [10] did an extensive research on H_f^ν on A_ν^2 ; see [2] for further results. For general symbols, Zhu [21] was the first to build up a bridge between Hankel operators and the mean oscillation of the symbols in the Bergman metric, and this idea has been further developed in several contexts [6–8,22]; see [23] and the references therein for further information on the theory of Hankel operators. More recently, Pau et al. [12] described the complex valued symbols f such that the Hankel operators H_f^β and $H_{\bar{f}}^\beta$ are simultaneously bounded from A_α^p to L_β^q , provided $1 < p \leq q < \infty$. Our primary aim is to extend this last-mentioned result to the context of radial B_q -weights. To do this, some definitions are needed. For a radial weight ω , we assume throughout the paper that $\widehat{\omega}(z) = \int_{|s|=|z|}^1 \omega(s) ds > 0$ for all $z \in \mathbb{D}$, for otherwise the Bergman space A_ω^p would contain

all analytic functions in \mathbb{D} . A radial weight ω belongs to the class $\widehat{\mathcal{D}}$ if there exists a constant $C = C(\omega) > 1$ such that $\widehat{\omega}(r) \leq C\widehat{\omega}(\frac{1+r}{2})$ for all $0 \leq r < 1$. Moreover, if there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that

$$\widehat{\omega}(r) \geq C\widehat{\omega}\left(1 - \frac{1-r}{K}\right), \quad 0 \leq r < 1, \quad (1.1)$$

then $\omega \in \check{\mathcal{D}}$. We write $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$ for short. For basic properties of these classes of weights and more, see [13, 14] and the references therein. Let $\beta(z, \zeta)$ denote the hyperbolic distance between $z, \zeta \in \mathbb{D}$, $\Delta(z, r)$ the hyperbolic disc of center z and radius $r > 0$, and $S(z)$ the Carleson square associated to z . For $0 < p, q < \infty$ and radial weights ω, ν , define

$$\gamma(z) = \gamma_{\omega, \nu, p, q}(z) = \frac{\widehat{\nu}(z)^{\frac{1}{q}}(1 - |z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}}(1 - |z|)^{\frac{1}{p}}}, \quad z \in \mathbb{D}. \quad (1.2)$$

Further, for $f \in L^1_{\nu, \text{loc}}$, write $\widehat{f}_{r, \nu}(z) = \frac{\int_{\Delta(z, r)} f(\zeta) \nu(\zeta) dA(\zeta)}{\nu(\Delta(z, r))}$ and

$$\text{MO}_{\nu, q, r}(f)(z) = \left(\frac{1}{\nu(\Delta(z, r))} \int_{\Delta(z, r)} |f(\zeta) - \widehat{f}_{r, \nu}(z)|^q \nu(\zeta) dA(\zeta) \right)^{\frac{1}{q}}$$

for all $z \in \mathbb{D}$. It is worth noticing that for prefixed $r > 0$, the quantity $\nu(\Delta(z, r))$ may equal to zero for some z arbitrarily close to the boundary if $\nu \in \widehat{\mathcal{D}}$. However, if $\nu \in \mathcal{D}$, then there exists $r_0 = r_0(\nu) > 0$ such that $\nu(\Delta(z, r)) \asymp \nu(S(z)) > 0$ for all $z \in \mathbb{D}$ if $r \geq r_0$. The space $\text{BMO}(\Delta)_{\omega, \nu, p, q, r}$ consists of $f \in L^q_{\nu, \text{loc}}$ such that

$$\|f\|_{\text{BMO}(\Delta)_{\omega, \nu, p, q, r}} = \sup_{z \in \mathbb{D}} (\text{MO}_{\nu, q, r}(f)(z) \gamma(z)) < \infty.$$

We will show that if $\nu \in \mathcal{D}$, then $\text{BMO}(\Delta)_{\omega, \nu, p, q, r}$ does not depend on r for $r \geq r_0$. In this case, we simply write $\text{BMO}(\Delta)_{\omega, \nu, p, q}$. The main result of this study reads as follows and it will be proved in Sect. 5.

Theorem 1 *Let $1 < p \leq q < \infty$, $\omega \in \mathcal{D}$, $\nu \in B_q$ a radial weight and $f \in L^q_{\nu}$. Then $H^{\nu}_f, H^{\nu}_{\bar{f}} : A^p_{\omega} \rightarrow L^q_{\nu}$ are bounded if and only if $f \in \text{BMO}(\Delta)_{\omega, \nu, p, q}$.*

The approach employed in the proof of this result follows the guideline of [12, Theorem 4.1], however a good number of steps cannot be adapted straightforwardly and need substantial modifications. In Sect. 2 we prove some results concerning the classes of weights involved in this work and the boundedness of the Bergman projection P_{ν} , while in Sect. 3 we introduce and study two spaces of functions on \mathbb{D} . One of them is denoted as $\text{BA}(\Delta)_{\omega, \nu, p, q}$, and although its initial definition depends on r , it can be described in terms of an appropriate Berezin transform or simply observing that $f \in \text{BA}(\Delta)_{\omega, \nu, p, q}$ if and only the multiplication operator $M_f(g) = fg$ is bounded from A^p_{ω} to L^q_{ν} [15]. The second one, denoted by $\text{BO}(\Delta)_{\omega, \nu, p, q}$, consists of continuous functions on \mathbb{D} such that the oscillation in the Bergman metric is bounded in terms of the auxiliary function γ given in (1.2). We show that $f \in \text{BO}(\Delta)_{\omega, \nu, p, q}$ if and only if

$$|f(z) - f(\zeta)| \lesssim \|f\|_{\text{BO}(\Delta)_{\omega, \nu, p, q}} (1 + \beta(z, \zeta)) \Gamma_{\tau}(z, \zeta) \quad z, \zeta \in \mathbb{D},$$

where

$$\Gamma_{\tau}(z, \zeta) = \frac{\left(\frac{|1-\bar{z}\zeta|^2}{\max\{1-|z|^2, 1-|\zeta|^2\}} \right)^{\frac{1}{p} - \frac{\tau+1}{q}} \widehat{\omega} \left(1 - \frac{2|1-\bar{z}\zeta|^2}{\max\{1-|z|^2, 1-|\zeta|^2\}} \right)^{\frac{1}{p}}}{\min \left\{ \frac{\widehat{v}(z)}{(1-|z|)^{\tau}}, \frac{\widehat{v}(\zeta)}{(1-|\zeta|)^{\tau}} \right\}^{\frac{1}{q}}}, \quad z, \zeta \in \mathbb{D},$$

for an appropriate (small) constant $\tau = \tau(\omega, \nu) > 0$. If ω and ν are standard weights, then Γ_{τ} does not coincide with the function playing the corresponding role in [12, Lemma 3.2]; in the latter case the function is simpler in many aspects and does not depend on the additional parameter τ . Then, we show that

$$\text{BMO}(\Delta)_{\omega, \nu, p, q} = \text{BA}(\Delta)_{\omega, \nu, p, q} + \text{BO}(\Delta)_{\omega, \nu, p, q}. \quad (1.3)$$

In order to prove this decomposition, due to the complex nature of $\Gamma_{\tau}(z, \zeta)$, we are forced to split \mathbb{D} into several regions depending on z , establish sharp estimates for $\Gamma_{\tau}(z, \zeta)$ in each region and then apply properties of weights in \mathcal{D} . The identity (1.3) together with a description of the boundedness of the integral operator

$$T_{b,c}f(z) = \int_{\mathbb{D}} f(\zeta) \frac{(1-|\zeta|^2)^b}{(1-z\bar{\zeta})^c} dA(\zeta)$$

and its maximal counterpart from A_{ω}^p to L_{ν}^q , see Sect. 4 below, are key tools to prove that each $f \in \text{BMO}(\Delta)_{\omega, \nu, p, q}$ induces a bounded Hankel operator from A_{ω}^p to L_{ν}^q . Theorem 1 will be proved in Sect. 5.

Finally, in Sect. 6, as a byproduct of Theorem 1, we describe the analytic symbols such that $H_{\bar{f}}^p : A_{\omega}^p \rightarrow L_{\nu}^q$ is bounded. The space $\mathcal{B}_{d\gamma}$ consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}_{d\gamma}} = \sup_{z \in \mathbb{D}} |f'(z)|(1-|z|)^{\gamma}(z) + |f(0)| < \infty,$$

where γ is given by (1.2).

Theorem 2 *Let $1 < p \leq q < \infty$, $\omega \in \mathcal{D}$, $\nu \in B_q$ a radial weight and $f \in A_{\nu}^1$. Then $H_{\bar{f}}^p : A_{\omega}^p \rightarrow L_{\nu}^q$ is bounded if and only if $f \in \mathcal{B}_{d\gamma}$.*

Throughout the paper $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 < p < \infty$. Further, the letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \leq Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we will write $a \asymp b$.

2 Auxiliary results

For a radial weight ω , $K > 1$ and $0 \leq r < 1$, let $\rho_n^r = \rho_n^r(\omega, K)$ be defined by $\widehat{\omega}(\rho_n^r) = \widehat{\omega}(r)K^{-n}$ for all $n \in \mathbb{N} \cup \{0\}$. Write $\rho_n = \rho_n^0$ for short. For $x \geq 1$, write $\omega_x = \int_0^1 r^x \omega(r) dr$. Denote

$$\omega^{\star}(z) = \int_{|z|}^1 \log \frac{s}{|z|} \omega(s) s ds, \quad z \in \mathbb{D} \setminus \{0\}.$$

Throughout the proofs we will repeatedly use several basic properties of weights in the classes $\widehat{\mathcal{D}}$ and $\check{\mathcal{D}}$. For a proof of the first lemma, see [13, Lemma 2.1]; the second one can be proved by similar arguments.

Lemma A Let ω be a radial weight. Then the following statements are equivalent:

- (i) $\omega \in \widehat{\mathcal{D}}$;
- (ii) There exist $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that

$$\widehat{\omega}(r) \leq C \left(\frac{1-r}{1-t} \right)^\beta \widehat{\omega}(t), \quad 0 \leq r \leq t < 1;$$

- (iii) There exist $C = C(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ such that

$$\int_0^t \left(\frac{1-t}{1-s} \right)^\gamma \omega(s) ds \leq C \widehat{\omega}(t), \quad 0 \leq t < 1;$$

- (iv) There exists $\lambda = \lambda(\omega) \geq 0$ such that

$$\int_{\mathbb{D}} \frac{dA(z)}{|1 - \bar{\zeta}z|^{\lambda+1}} \asymp \frac{\widehat{\omega}(\zeta)}{(1 - |\zeta|)^\lambda}, \quad \zeta \in \mathbb{D};$$

- (v) There exist $K = K(\omega) > 1$ and $C = C(\omega, K) > 1$ such that $1 - \rho_n^r(\omega, K) \geq C(1 - \rho_{n+1}^r(\omega, K))$ for some (equivalently for all) $0 \leq r < 1$ and for all $n \in \mathbb{N} \cup \{0\}$.

Lemma B Let ω be a radial weight. Then $\omega \in \check{\mathcal{D}}$ if and only if there exist $C = C(\omega) > 0$ and $\alpha = \alpha(\omega) > 0$ such that

$$\widehat{\omega}(t) \leq C \left(\frac{1-t}{1-r} \right)^\alpha \widehat{\omega}(r), \quad 0 \leq r \leq t < 1.$$

Two more results on weights of more general nature than Lemmas A and B are also needed.

Lemma 3 Let ω be a radial weight. Then the following statements are equivalent:

- (i) $\omega \in \widehat{\mathcal{D}}$;
- (ii) For some (equivalently for each) $v \in \mathcal{D}$ there exists a constant $C = C(\omega, v) > 0$ such that

$$\int_r^1 \frac{\omega(t)\widehat{v}(t)}{\widehat{\omega}(t)} dt \leq C \widehat{v}(r), \quad 0 \leq r < 1;$$

- (iii) For some (equivalently for each) $v \in \mathcal{D}$ there exists a constant $C = C(\omega, v) > 0$ such that

$$\int_0^r \frac{\omega(t)}{\widehat{\omega}(t)\widehat{v}(t)} dt \leq \frac{C}{\widehat{v}(r)}, \quad 0 \leq r < 1.$$

Proof Let first $\omega \in \widehat{\mathcal{D}}$ and $0 \leq r < 1$, and consider $\rho_n^r = \rho_n^r(\omega, K)$ for all $n \in \mathbb{N} \cup \{0\}$. Then Lemma B, applied to $v \in \mathcal{D} \subset \check{\mathcal{D}}$, and Lemma A(v), applied to ω , imply

$$\begin{aligned} \int_r^1 \frac{\omega(t)\widehat{v}(t)}{\widehat{\omega}(t)} dt &= \sum_{n=0}^{\infty} \int_{\rho_n^r}^{\rho_{n+1}^r} \frac{\omega(t)\widehat{v}(t)}{\widehat{\omega}(t)} dt \leq \sum_{n=0}^{\infty} \widehat{v}(\rho_n^r) \int_{\rho_n^r}^{\rho_{n+1}^r} \frac{\omega(t)}{\widehat{\omega}(t)} dt \\ &\lesssim \log K \frac{\widehat{v}(\rho_0^r)}{(1 - \rho_0^r)^\beta} \sum_{n=0}^{\infty} (1 - \rho_n^r)^\beta \\ &\leq \widehat{v}(r) \log K \sum_{n=0}^{\infty} \frac{1}{(C^\beta)^n} = \widehat{v}(r) \log K \frac{C^\beta}{C^\beta - 1}, \quad 0 \leq r < 1, \end{aligned}$$

for a suitably fixed $K = K(\omega) > 1$, and thus (ii) is satisfied. Conversely, (ii) implies

$$C\widehat{v}(r) \geq \int_r^1 \frac{\omega(t)\widehat{v}(t)}{\widehat{\omega}(t)} dt \geq \int_r^{\frac{1+r}{2}} \frac{\omega(t)\widehat{v}(t)}{\widehat{\omega}(t)} dt \geq \widehat{v}\left(\frac{1+r}{2}\right) \log \frac{\widehat{\omega}(r)}{\widehat{\omega}\left(\frac{1+r}{2}\right)}, \quad 0 \leq r < 1,$$

and since $v \in \mathcal{D} \subset \widehat{\mathcal{D}}$ by the hypothesis, we deduce $\widehat{\omega}(r) \lesssim \widehat{\omega}\left(\frac{1+r}{2}\right)$ for all $0 \leq r < 1$. Thus $\omega \in \widehat{\mathcal{D}}$.

Let $\omega \in \widehat{\mathcal{D}}$ and $0 \leq r < 1$, and consider $\rho_n = \rho_n(\omega, K)$ for all $n \in \mathbb{N} \cup \{0\}$. Fix $k = k(\omega, K) \in \mathbb{N} \cup \{0\}$ such that $\rho_k \leq r < \rho_{k+1}$. Then

$$\int_0^r \frac{\omega(t)}{\widehat{\omega}(t)\widehat{v}(t)} dt = \sum_{n=0}^{k-1} \int_{\rho_n}^{\rho_{n+1}} \frac{\omega(t)}{\widehat{\omega}(t)\widehat{v}(t)} dt + \int_{\rho_k}^r \frac{\omega(t)}{\widehat{\omega}(t)\widehat{v}(t)} dt, \quad 0 \leq r < 1,$$

where, by Lemma B, applied to $v \in \mathcal{D} \subset \widetilde{\mathcal{D}}$, and Lemma A(v), applied to ω ,

$$\begin{aligned} \sum_{n=0}^{k-1} \int_{\rho_n}^{\rho_{n+1}} \frac{\omega(t)}{\widehat{\omega}(t)\widehat{v}(t)} dt &\leq \sum_{n=0}^{k-1} \frac{1}{\widehat{v}(\rho_{n+1})} \int_{\rho_n}^{\rho_{n+1}} \frac{\omega(t)}{\widehat{\omega}(t)} dt \\ &\lesssim \sum_{n=0}^{k-1} \frac{(1-\rho_k)^\alpha}{\widehat{v}(\rho_k)} \frac{1}{(1-\rho_{n+1})^\alpha} \log \left(\frac{\widehat{\omega}(\rho_n)}{\widehat{\omega}(\rho_{n+1})} \right) \\ &\leq \log K \frac{(1-\rho_k)^\alpha}{\widehat{v}(r)} \sum_{n=0}^{k-1} \frac{1}{(C^\alpha)^{k-1-n} (1-\rho_k)^\alpha} \\ &\leq \frac{\log K}{\widehat{v}(r)} \sum_{n=0}^{\infty} \frac{1}{(C^\alpha)^n} = \frac{\log K}{\widehat{v}(r)} \frac{C^\alpha}{C^\alpha - 1}, \quad k \in \mathbb{N}, \end{aligned}$$

for some $\alpha = \alpha(v) > 0$ and for a suitably fixed $K = K(\omega) > 1$, and similarly,

$$\int_{\rho_k}^r \frac{\omega(t)}{\widehat{\omega}(t)\widehat{v}(t)} dt \leq \frac{1}{\widehat{v}(r)} \log \left(\frac{\widehat{\omega}(\rho_k)}{\widehat{\omega}(r)} \right) \leq \frac{\log K}{\widehat{v}(r)}, \quad k \in \mathbb{N} \cup \{0\}.$$

The statement (iii) follows from these estimates.

Conversely, by replacing r by $\frac{1+r}{2}$ in (iii) we obtain

$$\frac{C}{\widehat{v}\left(\frac{1+r}{2}\right)} \geq \int_0^{(1+r)/2} \frac{\omega(t)}{\widehat{\omega}(t)\widehat{v}(t)} dt \geq \int_r^{(1+r)/2} \frac{\omega(t)}{\widehat{\omega}(t)\widehat{v}(t)} dt \geq \frac{1}{\widehat{v}(r)} \log \frac{\widehat{\omega}(r)}{\widehat{\omega}\left(\frac{1+r}{2}\right)}, \quad 0 \leq r < 1,$$

and since $v \in \mathcal{D} \subset \widehat{\mathcal{D}}$ by the hypothesis, we deduce $\widehat{\omega}(r) \lesssim \widehat{\omega}\left(\frac{1+r}{2}\right)$ for all $0 \leq r < 1$. Thus $\omega \in \widehat{\mathcal{D}}$. \square

Lemma 4 Let $\omega, v \in \mathcal{D}$, and denote $\sigma = \sigma_{\omega, v} = \omega\widehat{v}/\widehat{\omega}$. Then $\widehat{\sigma} \asymp \widehat{v}$ on $[0, 1)$, and hence $\sigma \in \mathcal{D}$.

Proof Lemma 3(ii) implies $\widehat{\sigma} \lesssim \widehat{v}$ on $[0, 1)$. The argument used to prove (i) \Rightarrow (ii) in the said lemma shows that $\widehat{\sigma} \gtrsim \widehat{v}$ on $[0, 1)$, provided $\omega \in \widetilde{\mathcal{D}}$ and $v \in \mathcal{D}$. Thus $\widehat{\sigma} \asymp \widehat{v}$, and $\sigma \in \mathcal{D}$ by Lemmas A(ii) and B. \square

The next lemma says that in many instances concerning A^p -norms we may replace ω by $\widetilde{\omega} = \widehat{\omega}/(1 - |\cdot|)$ if $\omega \in \mathcal{D}$. This result has the flavor of radial Carleson measures and indeed can be established by appealing to the characterization of Carleson measures for the Bergman space A_ω^p induced by $\omega \in \widehat{\mathcal{D}}$ given in [15]. That approach requires showing that the involved

weights belong to $\widehat{\mathcal{D}}$, which is of course the case, and thus involves more calculations than the simple proof given below.

Lemma 5 *Let $0 < p < \infty$, $\omega \in \mathcal{D}$ and $-\alpha < \kappa < \infty$, where $\alpha = \alpha(\omega) > 0$ is that of Lemma B. Then*

$$\int_{\mathbb{D}} |f(z)|^p (1 - |z|)^\kappa \omega(z) dA(z) \asymp \int_{\mathbb{D}} |f(z)|^p (1 - |z|)^{\kappa-1} \widehat{\omega}(z) dA(z), \quad f \in \mathcal{H}(\mathbb{D}). \quad (2.1)$$

Proof The function $(1 - |\cdot|)^{\kappa-1} \widehat{\omega}$ is a weight for each $\kappa > -\alpha$ by Lemma B. Therefore an integration by parts shows that (2.1) is equivalent to

$$\int_0^1 \frac{\partial}{\partial r} M_p^p(r, f) \left(\int_r^1 (1-t)^\kappa \omega(t) dt \right) dr \asymp \int_0^1 \frac{\partial}{\partial r} M_p^p(r, f) \left(\int_r^1 (1-t)^{\kappa-1} \widehat{\omega}(t) dt \right) dr.$$

Another integration by parts reveals that both integrals from r to 1 above are bounded by a constant times $\widehat{\omega}(r)(1-r)^\kappa$. But Lemma A(ii) implies

$$\int_r^1 (1-t)^{\kappa-1} \widehat{\omega}(t) dt \gtrsim \frac{\widehat{\omega}(r)}{(1-r)^{\beta(\omega)}} \int_r^1 (1-t)^{\kappa-1+\beta(\omega)} dt \asymp \widehat{\omega}(r)(1-r)^\kappa, \quad 0 \leq r < 1,$$

and

$$\int_r^1 (1-t)^\kappa \omega(t) dt \gtrsim \frac{\widehat{\omega}(r)}{(1-r)^{\beta(\omega)}} \int_r^1 \frac{\omega(t)(1-t)^{\kappa+\beta(\omega)}}{\widehat{\omega}(t)} dt \asymp \widehat{\omega}(r)(1-r)^\kappa, \quad 0 \leq r < 1,$$

by Lemma 4. The assertion follows. \square

The last auxiliary results shows that each radial weight in the Bekollé–Bonami class B_q belongs to \mathcal{D} , and for each $v \in \mathcal{D}$ the maximal Bergman projection

$$P_v^+(f)(z) = \int_{\mathbb{D}} f(\zeta) |B_z^v(\zeta)| v(\zeta) dA(\zeta), \quad z \in \mathbb{D},$$

is bounded on L_v^q . It is worth noticing that obviously $\mathcal{D} \not\subset \cup_{1 < q < \infty} B_q$ because $v \in \mathcal{D}$ may vanish on a set of positive measure.

Proposition 6 *Let $1 < q < \infty$ and $v \in B_q$ a radial weight. Then $v \in \mathcal{D}$. Moreover, $P_v^+ : L_v^q \rightarrow L_v^q$ is bounded for all $v \in \mathcal{D}$.*

Proof If $v \in B_q$, then by [5] there exists $\beta > -1$ such that

$$\left(\int_{S(a)} v(z) dA(z) \right)^{\frac{1}{q}} \left(\int_{S(a)} \left(\frac{(1-|z|)^\beta}{v(z)} \right)^{\frac{q'}{q}} (1-|z|)^\beta dA(z) \right)^{\frac{1}{q'}} \lesssim (1-|a|)^{(2+\beta)}, \quad a \in \mathbb{D}.$$

Since v is radial, this condition easily implies $v \in \mathcal{D}$.

Let now $1 < q < \infty$ and $v \in \mathcal{D}$, and define $h = \widehat{v}^{-\frac{1}{qq'}}$. Then $\int_t^1 h(s)^{q'} v(s) ds \asymp \widehat{v}(t)^{\frac{1}{q'}}$ for all $0 \leq t < 1$. Therefore Lemma B yields

$$\int_0^r \frac{\int_t^1 h(s)^{q'} v(s) ds}{\widehat{v}(t)(1-t)} dt \asymp \int_0^r \frac{dt}{\widehat{v}(t)^{\frac{1}{q}}(1-t)} \lesssim \frac{1}{\widehat{v}(r)^{\frac{1}{q}}} = h^{q'}(r), \quad 0 \leq r < 1. \quad (2.2)$$

Moreover, by symmetry, (2.2) with q' in place of q is satisfied. Since $v \in \widehat{\mathcal{D}}$, we may apply [16, Theorem 1] and (2.2) to deduce

$$\int_{\mathbb{D}} |B_z^v(\zeta)| h^{p'}(\zeta) v(\zeta) dA(\zeta) \lesssim h^{p'}(z), \quad z \in \mathbb{D},$$

and

$$\int_{\mathbb{D}} |B_z^v(\zeta)| h^p(z) v(\zeta) dA(\zeta) \lesssim h^p(z), \quad \zeta \in \mathbb{D}.$$

It follows from Schur's test [23, Theorem 3.6] that the maximal Bergman projection $P_v^+ : L_v^p \rightarrow L_v^p$ is bounded. \square

3 Some spaces of functions

Recall that

$$\gamma(z) = \gamma_{\omega, v, p, q}(z) = \frac{\widehat{v}(z)^{\frac{1}{q}} (1 - |z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}} (1 - |z|)^{\frac{1}{p}}}, \quad z \in \mathbb{D}, \quad (3.1)$$

and $\widehat{f}_{r, v}(z) = \frac{\int_{\Delta(z, r)} f(\zeta) v(\zeta) dA(\zeta)}{v(\Delta(z, r))}$ for $f \in L_{v, \text{loc}}^1$, and

$$\text{MO}_{v, q, r}(f)(z) = \left(\frac{1}{v(\Delta(z, r))} \int_{\Delta(z, r)} |f(\zeta) - \widehat{f}_{r, v}(z)|^q v(\zeta) dA(\zeta) \right)^{\frac{1}{q}}$$

for all $z \in \mathbb{D}$. If $v \in \widetilde{\mathcal{D}}$, then by the definition there exist $K = K(v) > 1$ and $C = C(v) > 1$ such that

$$\int_r^{1-\frac{1-r}{K}} v(s) ds \geq (C-1) \widehat{v} \left(1 - \frac{1-r}{K} \right) > 0, \quad 0 \leq r < 1.$$

It follows that there exists $r_v \in (0, \infty)$ such that $v(\Delta(z, r)) > 0$ for all $z \in \mathbb{D}$ if $r \geq r_v$.

The space $\text{BMO}(\Delta) = \text{BMO}(\Delta)_{\omega, v, p, q, r}$ consists of $f \in L_{v, \text{loc}}^q$ such that

$$\|f\|_{\text{BMO}(\Delta)} = \sup_{z \in \mathbb{D}} (\text{MO}_{v, q, r}(f)(z) \gamma(z)) < \infty.$$

The following lemma is easy to establish; see [12, Lemma 3.1] for a similar result.

Lemma 7 *Let $1 \leq p, q < \infty$, ω a radial weight, $v \in \widetilde{\mathcal{D}}$ and $r_v \leq r < \infty$. Then*

$$\text{MO}_{v, q, r}(f)(z) \leq 2 \left(\frac{1}{v(\Delta(z, r))} \int_{\Delta(z, r)} |f(\zeta) - \lambda|^q v(\zeta) dA(\zeta) \right)^{\frac{1}{q}}, \quad z \in \mathbb{D}, \lambda \in \mathbb{C}, f \in L_v^q,$$

and therefore $f \in L_v^q$ belongs to $\text{BMO}(\Delta)$ if and only if for each $z \in \mathbb{D}$ there exists $\lambda_z \in \mathbb{C}$ such that

$$\sup_{z \in \mathbb{D}} \left(\frac{\gamma(z)^q}{v(\Delta(z, r))} \int_{\Delta(z, r)} |f(\zeta) - \lambda_z|^q v(\zeta) dA(\zeta) \right) < \infty.$$

For $0 < p, q < \infty$, $0 \leq \tau < \infty$ and radial weights ω, ν , let

$$\Gamma_\tau(z, \zeta) = \frac{\left(\frac{|1 - \bar{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \right)^{\frac{1}{p} - \frac{\tau+1}{q}} \widehat{\omega} \left(1 - \frac{2|1 - \bar{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \right)^{\frac{1}{p}}}{\min \left\{ \frac{\widehat{\nu}(z)}{(1 - |z|)^\tau}, \frac{\widehat{\nu}(\zeta)}{(1 - |\zeta|)^\tau} \right\}^{\frac{1}{q}}}, \quad z, \zeta \in \mathbb{D}, \quad (3.2)$$

with the understanding that $\widehat{\omega}(t) = \widehat{\omega}(0)$ when $t < 0$. The following lemma explains the behavior of Γ_τ near the diagonal.

Lemma 8 *Let $0 < p, q, r < \infty$, $0 \leq \tau < \infty$ and $\omega, \nu \in \widehat{\mathcal{D}}$. Then*

$$\Gamma_\tau(z, \zeta) \asymp \gamma(z)^{-1} \asymp \gamma(\zeta)^{-1}, \quad \beta(z, \zeta) \leq r.$$

Proof Clearly

$$|1 - \bar{z}\zeta| \asymp 1 - |z| \asymp 1 - |\zeta|, \quad \beta(z, \zeta) \leq r,$$

and hence there exist $0 < m_r < 1 < M_r < \infty$ such that

$$m_r(1 - |z|) \leq \frac{2|1 - \bar{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \leq M_r(1 - |z|), \quad \beta(z, \zeta) \leq r.$$

Since $\omega \in \widehat{\mathcal{D}}$ by the hypothesis, and $\widehat{\omega}(t) = \widehat{\omega}(0)$ for $t < 0$, Lemma A(ii) implies

$$\widehat{\omega}(z) \leq \frac{C}{m_r^\beta} \widehat{\omega}(1 - m_r(1 - |z|)) \leq \frac{C}{m_r^\beta} \widehat{\omega} \left(1 - \frac{2|1 - \bar{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \right), \quad \beta(z, \zeta) \leq r,$$

and

$$\widehat{\omega} \left(1 - \frac{2|1 - \bar{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \right) \leq C M_r^\beta \widehat{\omega}(1 - M_r(1 - |z|)) \leq C M_r^\beta \widehat{\omega}(z), \quad \beta(z, \zeta) \leq r,$$

for some $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$. Further, $\widehat{\nu}(z) \asymp \widehat{\nu}(\zeta)$ and $\widehat{\omega}(z) \asymp \widehat{\omega}(\zeta)$ if $\beta(z, \zeta) \leq r$ by Lemma A(ii). The assertion follows from these estimates. \square

For continuous $f : \mathbb{D} \rightarrow \mathbb{C}$ and $0 < r < \infty$, define

$$\Omega_r f(z) = \sup \{|f(z) - f(\zeta)| : \beta(z, \zeta) < r\}, \quad z \in \mathbb{D},$$

and let $\text{BO}(\Delta) = \text{BO}(\Delta)_{\omega, \nu, p, q, r}$ denote the space of those f such that

$$\|f\|_{\text{BO}(\Delta)} = \sup_{z \in \mathbb{D}} (\Omega_r f(z) \gamma(z)) < \infty.$$

Lemma 9 shows that the space $\text{BO}(\Delta) = \text{BO}(\Delta)_{\omega, \nu, p, q, r}$ is independent of r .

Lemma 9 *Let $0 < p \leq q < \infty$, $0 < r < \infty$, $\omega, \nu \in \check{\mathcal{D}}$ and $\gamma(z) = \gamma_{\omega, \nu, p, q}(z) = \frac{\widehat{\nu}(z)^{\frac{1}{q}} (1 - |z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}} (1 - |z|)^{\frac{1}{p}}}$. Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be continuous, and $0 < \tau < \min\{q\alpha(\omega)/p, \alpha(\nu)\}$, where $\alpha(\nu)$ and $\alpha(\omega)$ are those from Lemma B. Then the following statements are equivalent:*

- (i) $f \in \text{BO}(\Delta)$;
- (ii) $|f(z) - f(\zeta)| \lesssim \|f\|_{\text{BO}(\Delta)} (1 + \beta(z, \zeta)) \Gamma_\tau(z, \zeta)$ for all $z, \zeta \in \mathbb{D}$.

Proof Lemma 8 shows that (ii) implies (i). For the converse, assume (i), that is,

$$|f(z) - f(\zeta)|\gamma(z) \leq \|f\|_{\text{BO}(\Delta)}, \quad \beta(z, \zeta) < r. \quad (3.3)$$

The estimate (ii) for $\beta(z, \zeta) \leq r$ then follows from Lemma 8. If $\beta(z, \zeta) > r$, let $N = \max\{n \in \mathbb{N} : n \leq \beta(z, \zeta)/r + 1\}$, and pick up $N + 1$ points from the geodesic joining z and ζ such that $\beta(z_j, z_{j+1}) = \beta(z, \zeta)/N < r$ for all $j = 0, \dots, N - 1$. Then, as the hyperbolic distance is additive along geodesics, (3.3) yields

$$|f(z) - f(\zeta)| \leq \sum_{j=0}^{N-1} |f(z_j) - f(z_{j+1})| \leq \|f\|_{\text{BO}(\Delta)} \sum_{j=0}^{N-1} \frac{\widehat{\omega}(z_j)}{\widehat{\nu}(z_j)} (1 - |z_j|)^{\frac{1}{p} - \frac{1}{q}}.$$

Next, observe that

$$1 - |z_j| \leq \frac{2|1 - \bar{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}}, \quad j = 0, \dots, N; \quad (3.4)$$

see the proof of [12, Lemma 3.2] for details. This together with the inequality $\frac{1}{p} - \frac{1}{q} \geq 0$ gives

$$\begin{aligned} |f(z) - f(\zeta)| &\leq \|f\|_{\text{BO}(\Delta)} \left(\frac{2|1 - \bar{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \right)^{\frac{1}{p} - \frac{1}{q}} \sum_{j=0}^{N-1} \frac{\widehat{\omega}(z_j)^{\frac{1}{p}}}{\widehat{\nu}(z_j)^{\frac{1}{q}}} \\ &= \|f\|_{\text{BO}(\Delta)} \left(\frac{2|1 - \bar{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \right)^{\frac{1}{p} - \frac{1}{q}} \sum_{j=0}^{N-1} \frac{\widehat{\omega}(z_j)^{\frac{1}{p}}}{(1 - |z_j|)^{\frac{\tau}{q}}} \frac{(1 - |z_j|)^{\frac{\tau}{q}}}{\widehat{\nu}(z_j)^{\frac{1}{q}}}. \end{aligned}$$

The election of τ together with Lemma B shows that the functions $\widehat{\omega}(r)/(1 - r)^{\frac{p\tau}{q}}$ and $\widehat{\nu}(r)/(1 - r)^{\tau}$ are essentially decreasing on $[0, 1)$. Therefore the inequalities (3.4) and $|z_j| \leq \max\{|z|, |\zeta|\}$ yield

$$\begin{aligned} |f(z) - f(\zeta)| &\lesssim \|f\|_{\text{BO}(\Delta)} \left(\frac{2|1 - \bar{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \right)^{\frac{1}{p} - \frac{\tau+1}{q}} \\ &\quad \cdot \widehat{\omega} \left(1 - \frac{2|1 - \bar{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \right)^{\frac{1}{p}} \sum_{j=0}^{N-1} \frac{(1 - |z_j|)^{\frac{\tau}{q}}}{\widehat{\nu}(z_j)^{\frac{1}{q}}} \\ &\lesssim \|f\|_{\text{BO}(\Delta)} \Gamma_{\tau}(z, \zeta) N \lesssim \|f\|_{\text{BO}(\Delta)} (1 + \beta(z, \zeta)) \Gamma_{\tau}(z, \zeta), \quad \beta(z, \zeta) > r. \end{aligned}$$

Therefore (ii) is satisfied. \square

For $0 < p, q < \infty$, $0 < r < \infty$ and radial weights ω, ν , the space $\text{BA}(\Delta) = \text{BA}(\Delta)_{\omega, \nu, p, q, r}$ consists of $f \in L^q_{\nu, \text{loc}}$ such that

$$\|f\|_{\text{BA}(\Delta)} = \sup_{z \in \mathbb{D}} \left(\left(\frac{1}{\nu(\Delta(z, r))} \int_{\Delta(z, r)} |f(\zeta)|^q \nu(\zeta) dA(\zeta) \right)^{\frac{1}{q}} \gamma(z) \right) < \infty.$$

For $c, \sigma \in \mathbb{R}$ and a radial weight ν , the general Berezin transform of $\varphi \in L^1_{\nu(1-|\cdot|)^{\sigma}}$ is defined by

$$B(\varphi)(z) = B_{\nu, c, \sigma}(\varphi)(z) = \frac{(1 - |z|^2)^{c+1}}{\widehat{\nu}(z)} \int_{\mathbb{D}} \varphi(\zeta) \frac{(1 - |\zeta|^2)^{\sigma}}{|1 - z\bar{\zeta}|^{2+c+\sigma}} \nu(\zeta) dA(\zeta), \quad z \in \mathbb{D}.$$

The next lemma shows, in particular, that the space $\text{BA}(\Delta) = \text{BA}(\Delta)_{\omega, v, p, q, r}$ is independent of r as long as r is sufficiently large depending on $v \in \mathcal{D}$.

Lemma 10 *Let $0 < p \leq q < \infty$, $0 < r < \infty$ and $\omega, v \in \mathcal{D}$, $\gamma(z) = \gamma_{\omega, v, p, q}(z) = \frac{\widehat{v}(z)^{\frac{1}{q}} (1-|z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}} (1-|z|)^{\frac{1}{p}}}$. If $f \in L^q_v$, then the following statements are equivalent:*

- (i) *There exists $r_0 = r_0(v) > 0$ such that $f \in \text{BA}(\Delta) = \text{BA}(\Delta)_{\omega, v, p, q, r}$ for all $r \geq r_0$;*
- (ii) *$|f|^q v dA$ is a q -Carleson measure for A^p_ω ;*
- (iii) *The identity operator $Id : A^p_\omega \rightarrow L^q_{|f|^q v}$ is bounded;*
- (iv) *The multiplication operator $M_f(g) = fg$ is bounded from A^p_ω to L^q_v ;*
- (v) *$\sup_{z \in \mathbb{D}} \gamma(z)^q B(|f|^q)(z) < \infty$ for all $\sigma > 1 - \frac{q}{p}(1 + \alpha)$ and $c > \max\{-1 - \sigma, \frac{q}{p}(1 + \beta) - 2\}$, where $\alpha = \alpha(\omega) > 0$ and $\beta = \beta(\omega) > 0$ are those of Lemmas A(ii) and B.*

Proof It is obvious that (ii), (iii) and (iv) are equivalent by the definitions. Assume (ii) is satisfied, that is,

$$\left(\int_{\mathbb{D}} |g(\zeta)|^q |f(\zeta)|^q v(\zeta) dA(\zeta) \right)^{\frac{1}{q}} \lesssim \|g\|_{A^p_\omega}, \quad g \in A^p_\omega. \quad (3.5)$$

For $z \in \mathbb{D}$, let $g_z(\zeta) = \left(\frac{1-|z|}{1-\bar{z}\zeta} \right)^{\frac{\lambda+1}{p}}$, where $\lambda = \lambda(\omega) > 0$ is that of Lemma A(iv). Further, since $v \in \check{\mathcal{D}}$ by the hypothesis, there exists $r_v \in (0, \infty)$ such that $v(\Delta(z, r)) > 0$ for all $r \geq r_v$. For $g = g_z$ and $r \geq r_v$, (3.5) yields

$$\left(\frac{1}{v(\Delta(z, r))} \int_{\Delta(z, r)} |f(\zeta)|^q v(\zeta) dA(\zeta) \right)^{\frac{1}{q}} \lesssim \frac{\|g_z\|_{A^p_\omega}}{v(\Delta(z, r))^{\frac{1}{q}}} \lesssim \frac{(\widehat{\omega}(z)(1-|z|))^{\frac{1}{p}}}{v(\Delta(z, r))^{\frac{1}{q}}}, \quad z \in \mathbb{D}.$$

But since $v \in \mathcal{D}$, applications of Lemmas A(ii) and B show that

$$v(\Delta(z, r)) \asymp \widehat{v}(z)(1-|z|), \quad z \in \mathbb{D}, \quad (3.6)$$

if r is sufficiently large. It follows that $f \in \text{BA}(\Delta) = \text{BA}(\Delta)_{\omega, v, p, q, r}$ for all such r , and thus (i) is satisfied.

Conversely, if (i) is satisfied, then by using (3.6) we deduce

$$\left(\int_{\Delta(z, r)} |f(\zeta)|^q v(\zeta) dA(\zeta) \right)^{\frac{1}{q}} \lesssim \widehat{\omega}(z)^{\frac{1}{p}} (1-|z|)^{\frac{1}{p}}, \quad z \in \mathbb{D}.$$

Therefore $|f|^q v dA$ is a q -Carleson measure for A^p_ω by [17, Theorem 3].

By integrating only over $\Delta(z, r)$ in (v) and using (3.6) we obtain (i) from (v). To complete the proof of the lemma, it remains to show the converse implication. To do this, pick up a sequence $\{a_j\}$ and $0 < r < \infty$ in accordance with [23, Lemma 4.7], and observe that $\widehat{\omega}$ is essentially constant in each hyperbolically bounded region by Lemma A(ii). Then by using (3.6), the hypothesis (i), the election of c and σ , and finally Lemmas A(ii) and B, we deduce

$$\begin{aligned}
\frac{\widehat{v}(z)B(|f|^q)(z)}{(1-|z|^2)^{c+1}} &\lesssim \sum_{j=1}^{\infty} \int_{\Delta(a_j,r)} |f(\zeta)|^q \frac{(1-|\zeta|^2)^\sigma}{|1-z\bar{\zeta}|^{2+c+\sigma}} v(\zeta) dA(\zeta) \\
&\lesssim \sum_{j=1}^{\infty} \frac{(1-|a_j|^2)^\sigma}{|1-z\bar{a}_j|^{2+c+\sigma}} \int_{\Delta(a_j,r)} |f(\zeta)|^q v(\zeta) dA(\zeta) \\
&\lesssim \sum_{j=1}^{\infty} \frac{(1-|a_j|^2)^{\sigma+1} \widehat{v}(a_j)}{|1-z\bar{a}_j|^{2+c+\sigma} v(\Delta(a_j,r))} \int_{\Delta(a_j,r)} |f(\zeta)|^q v(\zeta) dA(\zeta) \\
&\lesssim \sum_{i=1}^{\infty} \frac{(1-|a_i|^2)^{\sigma+1} \widehat{v}(a_i)}{|1-z\bar{a}_i|^{2+c+\sigma} \gamma(a_i)^q} \asymp \sum_{j=1}^{\infty} \frac{(1-|a_j|^2)^{\sigma+\frac{q}{p}} \widehat{\omega}(a_j)^{\frac{q}{p}}}{|1-z\bar{a}_j|^{2+c+\sigma}} \\
&\lesssim \int_{\mathbb{D}} \frac{(1-|u|^2)^{\sigma+\frac{q}{p}-2} \widehat{\omega}(u)^{\frac{q}{p}}}{|1-z\bar{u}|^{2+c+\sigma}} dA(u) \\
&\lesssim \int_0^{|z|} \frac{\widehat{\omega}(t)^{\frac{q}{p}}}{(1-t)^{c+3-\frac{q}{p}}} dt + \frac{1}{(1-|z|)^{c+\sigma+1}} \int_{|z|}^1 (1-t)^{\sigma+\frac{q}{p}-2} \widehat{\omega}(t)^{\frac{q}{p}} dt \\
&\lesssim \frac{\widehat{\omega}(|z|)^{\frac{q}{p}}}{(1-|z|)^{c+2-\frac{q}{p}}} \asymp \frac{\widehat{v}(z)}{(1-|z|^2)^{c+1} \gamma(z)^q}, \quad z \in \mathbb{D},
\end{aligned}$$

and thus (v) is satisfied. \square

With these preparations we are ready to show that $\text{BMO}(\Delta) = \text{BA}(\Delta) + \text{BO}(\Delta)$. This follows from the case (ii) of the next theorem.

Theorem 11 Let $1 \leq p \leq q < \infty$, $\omega, v \in \mathcal{D}$, $\gamma(z) = \gamma_{\omega,v,p,q}(z) = \frac{\widehat{v}(z)^{\frac{1}{q}}(1-|z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}}(1-|z|)^{\frac{1}{p}}}$ and $f \in L_v^q$. Further, let $r \geq r_v$, $\sigma > 0$ and

$$c > 2\frac{q}{p}(\beta(\omega) + 1) + \sigma + \max\{2\beta(v), \gamma(v)\},$$

where $\beta(\omega), \beta(v), \gamma(v) > 0$ are associated to v and ω via Lemma A(ii), (iii). Then the following statements are equivalent:

- (i) There exists $r_0 = r_0(v) \geq r_v$ such that $f \in \text{BMO}(\Delta) = \text{BMO}(\Delta)_{\omega,v,p,q,r}$ for all $r \geq r_0$;
- (ii) $f = f_1 + f_2$, where $f_1 \in \text{BA}(\Delta)$ and $f_2 = \widehat{f}_{r,v} \in \text{BO}(\Delta)$;
- (iii) $\sup_{z \in \mathbb{D}} (B(|f - \widehat{f}_{r,v}(z)|^q) \gamma(z)^q) < \infty$;
- (iv) For each $z \in \mathbb{D}$ there exists $\lambda_z \in \mathbb{C}$ such that $\sup_{z \in \mathbb{D}} (B(|f - \lambda_z|^q) \gamma(z)^q) < \infty$.

Proof Obviously, (iii) implies (iv). Next assume (iv). The relation (3.6) shows that there exists $r_0 = r_0(v) > 0$ such that

$$\begin{aligned}
&\frac{1}{v(\Delta(z,r))} \int_{\Delta(z,r)} |f(\zeta) - \lambda_z|^q v(\zeta) dA(\zeta) \\
&\lesssim \frac{(1-|z|)^{c+1}}{\widehat{v}(z)} \int_{\mathbb{D}} |f(\zeta) - \lambda_z|^q \frac{(1-|\zeta|^2)^\sigma}{|1-z\bar{\zeta}|^{2+c+\sigma}} v(\zeta) dA(\zeta), \quad z \in \mathbb{D}, \quad r_0 \leq r < \infty,
\end{aligned}$$

which together with Lemma 7 shows that (i) is satisfied.

Assume now (i), and let $f_2 = \widehat{f}_{r,v}$. Since $f \in L_v^q$, $q \geq 1$ and $r \geq r_v$, the function f_2 is well defined and continuous. Since $\omega, v \in \mathcal{D}$ by the hypothesis, one may use Lemmas A(ii)

and **B** together with the argument in [12, 1651–1652] with minor modifications to show that $f_2 = \widehat{f}_{r,v} \in \text{BO}(\Delta)$ and $f_1 = f - \widehat{f}_{r,v} \in \text{BA}(\Delta)$. Thus (ii) is satisfied.

To complete the proof it suffices to show that (ii) implies (iii), so assume $f = f_1 + f_2$, where $f_1 \in \text{BA}(\Delta)$ and $f_2 = \widehat{f}_{r,v} \in \text{BO}(\Delta)$. Since $\widehat{f}_{r,v} = \widehat{f}_{1r,v} + \widehat{f}_{2r,v}$, it suffices to prove the condition in (iii) for f_1 and f_2 separately. First observe that by Lemma **A**(iii) the constant function 1 satisfies

$$B(1)(z) \lesssim \frac{(1-|z|)^{c+1}}{\widehat{v}(z)} \left(\int_0^{|z|} \frac{v(t)}{(1-t)^{1+c}} dt + \frac{1}{(1-|z|)^{1+c+\sigma}} \int_{|z|}^1 (1-t)^\sigma v(t) dt \right) \lesssim 1, \quad z \in \mathbb{D},$$

because $c > \max\{\gamma(v), \sigma\} - 1$ by the hypothesis. This together with Hölder's inequality and Lemma **10** yields

$$\begin{aligned} B(|f_1 - \widehat{f}_{1r,v}(z)|^q) \gamma(z)^q &\lesssim (B(|f_1|^q)(z) + |\widehat{f}_{1r,v}(z)|^q) \gamma(z)^q \\ &\leq (B(|f_1|^q)(z) + |\widehat{f}_{1r,v}(z)|^q) \gamma(z)^q \lesssim 1, \quad z \in \mathbb{D}, \end{aligned}$$

and thus (iii) for $f_1 \in \text{BA}(\Delta)$ is satisfied.

To deal with $f_2 \in \text{BO}(\Delta)$, pick up τ satisfying the hypothesis of Lemma **9**. Then

$$\begin{aligned} |f_2(\zeta) - \widehat{f}_{2r,v}(z)| &= \left| \frac{1}{v(\Delta(z,r))} \int_{\Delta(z,r)} (f_2(\zeta) - f_2(u)) v(u) dA(u) \right| \\ &\leq \frac{1}{v(\Delta(z,r))} \int_{\Delta(z,r)} |f_2(\zeta) - f_2(u)| v(u) dA(u) \\ &\lesssim \frac{1}{v(\Delta(z,r))} \int_{\Delta(z,r)} (1 + \beta(\zeta, u)) \Gamma_\tau(\zeta, u) v(u) dA(u) \\ &\lesssim (1 + \beta(z, \zeta)) \Gamma_\tau(z, \zeta), \quad z, \zeta \in \mathbb{D}, \end{aligned}$$

because $\Gamma_\tau(\zeta, u) \asymp \Gamma_\tau(z, \zeta)$ for all $u \in \Delta(z, r)$ by Lemma **A**(ii); see the proof of Lemma **8** for similar estimates. Hence it suffices to show that

$$\frac{(1-|z|)^{c+1} \gamma(z)^q}{\widehat{v}(z)} \int_{\mathbb{D}} |(1 + \beta(z, \zeta)) \Gamma_\tau(z, \zeta)|^q \frac{(1-|\zeta|^2)^\sigma}{|1-z\bar{\zeta}|^{2+c+\sigma}} v(\zeta) dA(\zeta) \lesssim 1, \quad z \in \mathbb{D}, \quad (3.7)$$

to obtain (iii) for $f_2 \in \text{BO}(\Delta)$. The proof of (3.7) is involved and will be divided into four separate cases. Before dealing with each case, we observe that since $\beta(z, \zeta)$ grows logarithmically, we may pick up $0 < \delta < \min\left\{\sigma, \frac{q}{p}\beta(\omega) + \beta(v) + \frac{\sigma}{2}\right\}$ and a constant $C = C(\delta) > 0$ such that

$$1 + \beta(z, \zeta) \leq C |(1 - |\varphi_z(\zeta)|)|^{-\frac{\delta}{q}} = C \left(\frac{|1 - \bar{z}\zeta|^2}{(1-|z|)(1-|\zeta|)} \right)^{\frac{\delta}{q}}, \quad z, \zeta \in \mathbb{D}. \quad (3.8)$$

Case 1 If

$$\zeta \in D_1(z) = \left\{ w \in \mathbb{D} : 1 - \frac{2|1 - z\bar{w}|^2}{1 - |z|^2} \leq 0 \right\},$$

then $1 - |z| \lesssim |1 - z\bar{\zeta}|^2$ and

$$\begin{aligned} \Gamma_\tau(z, \zeta)^q &\leq \frac{\left(\frac{|1 - z\bar{\zeta}|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}}\right)^{\frac{q}{p} - \tau - 1}}{\min\left\{\frac{\widehat{v}(z)}{(1 - |z|)^\tau}, \frac{\widehat{v}(\zeta)}{(1 - |\zeta|)^\tau}\right\}} \widehat{\omega}(0)^{\frac{q}{p}} \lesssim \left(\frac{|1 - z\bar{\zeta}|^2}{1 - |z|^2}\right)^{\frac{q}{p} - \tau - 1} \frac{(1 - |z|)^\tau}{\widehat{v}(z)} \chi_{D(0, |z|)}(\zeta) \\ &\quad + \left(\frac{|1 - z\bar{\zeta}|^2}{1 - |z|^2}\right)^{\frac{q}{p} - \tau - 1} \frac{(1 - |\zeta|)^\tau}{\widehat{v}(\zeta)} \chi_{\mathbb{D} \setminus D(0, |z|)}(\zeta), \quad z \in \mathbb{D}, \quad \zeta \in D_1(z), \end{aligned}$$

because of how τ is chosen in Lemma 9. Therefore (3.8) together with Lemmas A(ii) and 3 (ii) yields

$$\begin{aligned} &\frac{(1 - |z|)^{c+1} \gamma(z)^q}{\widehat{v}(z)} \int_{D_1(z)} |(1 + \beta(z, \zeta)) \Gamma_\tau(z, \zeta)|^q \frac{(1 - |\zeta|^2)^\sigma}{|1 - z\bar{\zeta}|^{2+c+\sigma}} v(\zeta) dA(\zeta) \\ &\lesssim \frac{(1 - |z|)^{c+2+2\tau-\delta-\frac{q}{p}} \gamma(z)^q}{\widehat{v}(z)^2} \int_{D_1(z) \cap D(0, |z|)} \frac{(1 - |\zeta|^2)^{\sigma-\delta}}{|1 - z\bar{\zeta}|^{4+c+\sigma-2(\frac{q}{p}+\delta-\tau)}} v(\zeta) dA(\zeta) \\ &\quad + \frac{(1 - |z|)^{c+2+\tau-\delta-\frac{q}{p}} \gamma(z)^q}{\widehat{v}(z)} \int_{D_1(z) \setminus D(0, |z|)} \frac{(1 - |\zeta|^2)^{\sigma-\delta+\tau}}{\widehat{v}(\zeta) |1 - z\bar{\zeta}|^{4+c+\sigma-2(\frac{q}{p}+\delta-\tau)}} v(\zeta) dA(\zeta) \\ &\lesssim \frac{(1 - |z|)^{\frac{c}{2}+\tau-\frac{q}{2}} \gamma(z)^q}{\widehat{v}(z)^2} \int_0^{|z|} (1 - s)^{\sigma-\delta} v(s) ds \\ &\quad + \frac{(1 - |z|)^{\frac{c}{2}-\frac{q}{2}} \gamma(z)^q}{\widehat{v}(z)} \int_{|z|}^1 (1 - s)^{\sigma-\delta+\tau} \frac{v(s)}{\widehat{v}(s)} ds \\ &\lesssim \frac{(1 - |z|)^{\frac{c}{2}+\tau-\frac{q}{2}+1-\frac{q}{p}}}{\widehat{v}(z) \widehat{\omega}(z)^{\frac{q}{p}}} + \frac{(1 - |z|)^{\frac{c}{2}+\frac{q}{2}+1+\tau-\delta-\frac{q}{p}}}{\widehat{\omega}(z)^{\frac{q}{p}}} \\ &\lesssim (1 - |z|)^{\frac{c}{2}+\tau-\frac{q}{2}+1-\frac{q}{p}-\beta(v)-\frac{q}{p}\beta(\omega)} \lesssim 1, \quad z \in \mathbb{D}, \end{aligned}$$

where the last estimate is an immediate consequence of the choices of c and δ .

Case 2 If

$$\zeta \in D_2(z) = \left\{ w \in \mathbb{D} : 1 - \frac{2|1 - z\bar{w}|^2}{1 - |z|^2} \geq |z| \geq |w| \right\},$$

then $|1 - z\bar{\zeta}| \asymp 1 - |z|^2 \leq 1 - |\zeta|^2$, which together the fact that $\frac{\widehat{v}(r)}{(1-r)^\tau}$ and $\frac{\widehat{\omega}(r)}{(1-r)^{\frac{p}{q}}}$ are essentially decreasing on $[0, 1)$ gives

$$\Gamma_\tau(z, \zeta)^q \lesssim \gamma(z)^{-q}, \quad z \in \mathbb{D}, \quad \zeta \in D_2(z).$$

Therefore (3.8) and Lemma A(iii) yield

$$\begin{aligned} &\frac{(1 - |z|)^{c+1} \gamma(z)^q}{\widehat{v}(z)} \int_{D_2(z)} |(1 + \beta(z, \zeta)) \Gamma_\tau(z, \zeta)|^q \frac{(1 - |\zeta|^2)^\sigma}{|1 - z\bar{\zeta}|^{2+c+\sigma}} v(\zeta) dA(\zeta) \\ &\lesssim \frac{(1 - |z|)^{c+1-\delta}}{\widehat{v}(z)} \int_{D_2(z)} \frac{(1 - |\zeta|^2)^{\sigma-\delta}}{|1 - z\bar{\zeta}|^{2+c+\sigma-2\delta}} v(\zeta) dA(\zeta) \\ &\lesssim \frac{(1 - |z|)^{c+1-\delta}}{\widehat{v}(z)} \int_0^{|z|} \frac{v(r)}{(1 - r)^{c+1-\delta}} dr \lesssim 1, \quad z \in \mathbb{D}. \end{aligned}$$

Case 3 If

$$\zeta \in D_3(z) = \left\{ w \in \mathbb{D} : \min \left\{ 1 - \frac{2|1 - z\bar{w}|^2}{1 - |z|^2}, |w| \right\} \geq |z| \right\},$$

then $|1 - z\bar{\zeta}| \asymp 1 - |z|^2 \geq 1 - |\zeta|^2$, which together the fact that $\frac{\widehat{v}(t)}{(1-t)^\tau}$ and $\frac{\widehat{\omega}(r)}{(1-r)^\tau \frac{p}{q}}$ are essentially decreasing on $[0, 1)$ implies

$$\Gamma_\tau(z, \zeta)^q \lesssim \frac{\widehat{\omega}(z)^{\frac{q}{p}} (1 - |z|)^{\frac{q}{p}-1}}{\widehat{v}(\zeta)}, \quad z \in \mathbb{D}, \quad \zeta \in D_3(z).$$

Therefore (3.8) and Lemma 3(ii) imply

$$\begin{aligned} & \frac{(1 - |z|)^{c+1} \gamma(z)^q}{\widehat{v}(z)} \int_{D_3(z)} |(1 + \beta(z, \zeta)) \Gamma_\tau(z, \zeta)|^q \frac{(1 - |\zeta|^2)^\sigma}{|1 - z\bar{\zeta}|^{2+c+\sigma}} v(\zeta) dA(\zeta) \\ & \lesssim (1 - |z|)^{c+1-\delta} \int_{D_3(z)} \frac{(1 - |\zeta|^2)^{\sigma-\delta}}{|1 - z\bar{\zeta}|^{2+c+\sigma-2\delta}} \frac{v(\zeta)}{\widehat{v}(\zeta)} dA(\zeta) \\ & \lesssim (1 - |z|)^{\delta-\sigma} \int_{|z|}^1 \frac{(1-s)^{\sigma-\delta} v(s)}{\widehat{v}(s)} ds \lesssim 1, \quad z \in \mathbb{D}. \end{aligned}$$

Case 4 If

$$\zeta \in D_4(z) = \left\{ w \in \mathbb{D} : 1 - \frac{2|1 - z\bar{w}|^2}{1 - |z|^2} < |z| \right\},$$

then Lemma A(ii) gives

$$\widehat{\omega} \left(1 - \frac{2|1 - z\bar{\zeta}|^2}{1 - |z|^2} \right) \lesssim \left(\frac{|1 - z\bar{\zeta}|}{1 - |z|} \right)^{2\beta(\omega)} \widehat{\omega}(z), \quad z \in \mathbb{D}, \quad \zeta \in D_4(z),$$

and hence

$$\begin{aligned} \Gamma_\tau(z, \zeta)^q & \lesssim \left(\frac{|1 - z\bar{\zeta}|}{1 - |z|} \right)^{2\beta(\omega) \frac{q}{p}} \widehat{\omega}(z)^{\frac{q}{p}} \left(\left(\frac{|1 - z\bar{\zeta}|^2}{1 - |z|} \right)^{\frac{q}{p}-\tau-1} \frac{(1 - |z|)^\tau}{\widehat{v}(z)} \chi_{D(0, |z|)}(\zeta) \right. \\ & \quad \left. + \left(\frac{|1 - z\bar{\zeta}|^2}{1 - |z|} \right)^{\frac{q}{p}-\tau-1} \frac{(1 - |\zeta|)^\tau}{\widehat{v}(\zeta)} \chi_{\mathbb{D} \setminus D(0, |z|)}(\zeta) \right), \quad z \in \mathbb{D}, \quad \zeta \in D_4(z). \end{aligned}$$

Therefore (3.8) and Lemmas A(iii) and 3 (ii) yield

$$\begin{aligned} & \frac{(1 - |z|)^{c+1} \gamma(z)^q}{\widehat{v}(z)} \int_{D_4(z)} |(1 + \beta(z, \zeta)) \Gamma_\tau(z, \zeta)|^q \frac{(1 - |\zeta|^2)^\sigma}{|1 - z\bar{\zeta}|^{2+c+\sigma}} v(\zeta) dA(\zeta) \\ & \lesssim \frac{(1 - |z|)^{c+2-\delta-\frac{q}{p}-2\beta(\omega)\frac{q}{p}+\tau}}{\widehat{v}(z)} \int_{D_4(z) \cap D(0, |z|)} \frac{(1 - |\zeta|^2)^{\sigma-\delta-\frac{q}{p}+\tau+1}}{|1 - z\bar{\zeta}|^{4+c+\sigma-2\delta-2\beta(\omega)\frac{q}{p}-2\frac{q}{p}+2\tau}} v(\zeta) dA(\zeta) \\ & \quad + (1 - |z|)^{c+2-\delta-\frac{q}{p}-2\beta(\omega)\frac{q}{p}-\frac{q}{p}+\tau+1} \int_{D_4(z) \setminus D(0, |z|)} \frac{(1 - |\zeta|)^{\sigma-\delta+\tau}}{|1 - z\bar{\zeta}|^{4+c+\sigma-2\delta-2\beta(\omega)\frac{q}{p}-2\frac{q}{p}+2\tau}} \frac{v(\zeta)}{\widehat{v}(\zeta)} dA(\zeta) \\ & \lesssim \frac{(1 - |z|)^{c+2-\delta-\frac{q}{p}-2\beta(\omega)\frac{q}{p}+\tau}}{\widehat{v}(z)} \int_0^{|z|} \frac{v(r)}{(1-r)^{2+c-\delta-2\beta(\omega)\frac{q}{p}-\frac{q}{p}+\tau}} dr \\ & \quad + \frac{1}{(1 - |z|)^{\sigma-\delta+\tau}} \int_{|z|}^1 \frac{(1-r)^{\sigma-\delta+\tau} v(r)}{\widehat{v}(r)} dr \lesssim 1, \quad z \in \mathbb{D}. \end{aligned}$$

Since $\mathbb{D} = \cup_{j=1}^4 D_j(z)$ for each $z \in \mathbb{D}$, by combining the four cases we obtain (3.7). Thus (ii) implies (iii), and the proof is complete. \square

4 Boundedness of integral operators

In order to deal with the boundedness of Hankel operators, we need a technical result concerning certain integral operators. For $f \in L_b^1$ and $b, c \in \mathbb{R}$, define

$$T_{b,c}(f)(z) = \int_{\mathbb{D}} f(\zeta) \frac{(1 - |\zeta|^2)^b}{(1 - z\bar{\zeta})^c} dA(\zeta), \quad z \in \mathbb{D},$$

and

$$S_{b,c}(f)(z) = \int_{\mathbb{D}} f(\zeta) \frac{(1 - |\zeta|^2)^b}{|1 - z\bar{\zeta}|^c} dA(\zeta), \quad z \in \mathbb{D}.$$

In the analytic case the operator $T_{b,c}$ can be interpreted as a fractional differentiation or integration depending on the parameters b and c [20]. The boundedness of these operator between L^p spaces induced by standard weights has been characterized in [19].

Lemma A(ii) shows that for $\eta \in \widehat{\mathcal{D}}$ there exists a constant $c_0 = c_0(\sigma) > 1$ such that hypotheses (i) and (ii) of the next lemma are satisfied for all $c \geq c_0$.

Lemma 12 *Let $1 < p \leq q < \infty$, $b > -1$, $c > 1$ and $\sigma, \eta \in \mathcal{D}$ such that*

- (i) $\int_r^1 \frac{(1-t)^{c-2}}{\widehat{\eta}(t)^{\frac{1}{q}}} dt \lesssim \frac{(1-r)^{c-1}}{\widehat{\eta}(r)^{\frac{1}{q}}}, \quad 0 \leq r < 1;$
- (ii) $\int_0^r \frac{\eta(t)}{(1-t)^{\frac{cq}{p}-1} \widehat{\eta}(t)^{\frac{1}{p'}}} dt \lesssim \frac{\widehat{\eta}(r)^{\frac{1}{p}}}{(1-r)^{\frac{cq}{p}-1}}, \quad 0 \leq r < 1.$

Then the following statements are equivalent:

1. $S_{b,c} : A_{\sigma}^p \rightarrow L_{\eta}^q$ is bounded;
2. $T_{b,c} : A_{\sigma}^p \rightarrow L_{\eta}^q$ is bounded;
3. $\sup_{0 < r < 1} (1-r)^{2+b-c+\frac{1}{q}-\frac{1}{p}} \frac{\widehat{\eta}(r)^{\frac{1}{q}}}{\widehat{\sigma}(r)^{\frac{1}{p}}} < \infty.$

Proof Obviously (1) implies (2). Assume now (2), and for each $\zeta \in \mathbb{D}$ and $N \in \mathbb{N}$, define $f_{\zeta,N} \in H^{\infty}$ by $f_{\zeta,N}(z) = \frac{z^N}{\sigma(S(\zeta))^{\frac{1}{p}}} \left(\frac{1-|\zeta|^2}{1-\bar{\zeta}z} \right)^{2+b+N}$ for all $z \in \mathbb{D}$. By differentiating the reproducing formula of A_b^2 we obtain

$$g^{(N)}(z) = M_1 \int_{\mathbb{D}} \frac{\bar{u}^N g(u) (1 - |u|^2)^b}{(1 - \bar{u}z)^{2+b+N}} dA(u), \quad z \in \mathbb{D}, \quad N \in \mathbb{N}, \quad g \in A_b^2, \quad (4.1)$$

where $M_1 = M_1(N, b) > 0$ is a constant. Therefore

$$\begin{aligned} T_{b,c}(f_{\zeta,N})(z) &= \frac{(1 - |\zeta|^2)^{2+b+N}}{\sigma(S(\zeta))^{\frac{1}{p}}} \int_{\mathbb{D}} \frac{u^N (1 - |u|^2)^b}{(1 - u\bar{\zeta})^{2+b+N} (1 - \bar{u}z)^c} dA(u) \\ &= \frac{(1 - |\zeta|^2)^{2+b+N}}{\sigma(S(\zeta))^{\frac{1}{p}}} \int_{\mathbb{D}} \frac{\bar{u}^N (1 - |u|^2)^b}{(1 - \zeta\bar{u})^{2+b+N} (1 - \bar{z}u)^c} dA(u) \\ &= M_2 \frac{(1 - |\zeta|^2)^{2+b+N}}{\sigma(S(\zeta))^{\frac{1}{p}}} \frac{z^N}{(1 - z\bar{\zeta})^{c+N}}, \end{aligned}$$

where $M_2 = M_2(b, c, N) > 0$. Fix $N > \max \left\{ \frac{\lambda(\eta)+1}{q} - c, \frac{\lambda(\sigma)+1}{p} - b - 2 \right\}$. Then Lemma A(iv) gives $\|f_{\zeta, N}\|_{L^p_\sigma} \asymp 1$ and

$$\int_{\mathbb{D}} \frac{\eta(z)}{|1 - \bar{\zeta}z|^{(c+N)q}} dA(z) \asymp \frac{\eta(S(\zeta))}{(1 - |\zeta|)^{(c+N)q}}, \quad \zeta \in \mathbb{D}.$$

Therefore (2) yields

$$\begin{aligned} \infty > \|f_{\zeta, N}\|_{L^p_\sigma}^q &\gtrsim \|T_{b,c}(f_{\zeta, N})\|_{L^q_\eta}^q \asymp \left(\frac{(1 - |\zeta|^2)^{2+b+N}}{\sigma(S(\zeta))^{\frac{1}{p}}} \right)^q \int_{\mathbb{D}} \frac{\eta(z)}{|1 - \bar{\zeta}z|^{(c+N)q}} dA(z) \\ &\asymp (1 - |\zeta|^2)^{q(2+b-c)} \frac{\eta(S(\zeta))}{\sigma(S(\zeta))^{\frac{q}{p}}}, \quad \zeta \in \mathbb{D}, \end{aligned}$$

thus (3) holds.

Assume (3) holds and let $h(\zeta) = \widehat{\sigma}(\zeta)^{\frac{1}{pp'}} (1 - |\zeta|^2)^{\frac{b}{p} + (\frac{1}{p} - \frac{1}{q})\frac{1}{p'}}$ for all $\zeta \in \mathbb{D}$. Then Hölder's inequality yields

$$\begin{aligned} |S_{b,c}f(z)| &\leq \left(\int_{\mathbb{D}} |f(\zeta)|^p h(\zeta)^p \frac{dA(\zeta)}{|1 - z\bar{\zeta}|^c} \right)^{\frac{1}{p}} \left(\int_{\mathbb{D}} \left(\frac{(1 - |\zeta|^2)^b}{h(\zeta)} \right)^{p'} \frac{dA(\zeta)}{|1 - z\bar{\zeta}|^c} \right)^{\frac{1}{p'}} \\ &= I_1(z)^{\frac{1}{p}} \cdot I_2(z)^{\frac{1}{p'}}, \end{aligned}$$

where

$$\begin{aligned} I_2(z) &= \int_{\mathbb{D}} \frac{(1 - |\zeta|^2)^{b - \frac{1}{p} + \frac{1}{q}}}{|1 - z\bar{\zeta}|^c \widehat{\sigma}(\zeta)^{\frac{1}{p}}} dA(\zeta) \asymp \int_0^1 \frac{(1 - r)^{b - \frac{1}{p} + \frac{1}{q}}}{\widehat{\sigma}(r)^{\frac{1}{p}} (1 - r|z|)^{c-1}} dr \\ &= \int_0^{|z|} \frac{(1 - r)^{b - \frac{1}{p} + \frac{1}{q}}}{\widehat{\sigma}(r)^{\frac{1}{p}} (1 - r|z|)^{c-1}} dr + \int_{|z|}^1 \frac{(1 - r)^{b - \frac{1}{p} + \frac{1}{q}}}{\widehat{\sigma}(r)^{\frac{1}{p}} (1 - r|z|)^{c-1}} dr = J^{|z|} + J_{|z|}. \end{aligned}$$

Lemma B together with the assumption (3) yields

$$J^{|z|} \leq \int_0^{|z|} \frac{(1 - r)^{b - \frac{1}{p} + \frac{1}{q} + 1 - c}}{\widehat{\sigma}(r)^{\frac{1}{p}}} dr \lesssim \int_0^{|z|} \frac{dr}{\widehat{\eta}(r)^{\frac{1}{q}} (1 - r)} \lesssim \frac{1}{\widehat{\eta}(z)^{\frac{1}{q}}}, \quad z \in \mathbb{D},$$

since $\eta \in \mathcal{D} \subset \check{\mathcal{D}}$ by the hypothesis. In a similar fashion, (3) together with the hypothesis (i) gives

$$J_{|z|} \leq \frac{1}{(1 - |z|)^{c-1}} \int_{|z|}^1 \frac{(1 - r)^{b - \frac{1}{p} + \frac{1}{q}}}{\widehat{\sigma}(r)^{\frac{1}{p}}} dr \lesssim \frac{1}{(1 - |z|)^{c-1}} \int_{|z|}^1 \frac{(1 - r)^{c-2}}{\widehat{\eta}(r)^{\frac{1}{q}}} dr \lesssim \frac{1}{\widehat{\eta}(z)^{\frac{1}{q}}}, \quad z \in \mathbb{D},$$

and hence $I_2(z) \lesssim \widehat{\eta}(z)^{-\frac{1}{q}}$ for all $z \in \mathbb{D}$. This estimate and Minkowski's integral inequality (Fubini's theorem in the case $q = p$) now yield

$$\begin{aligned} \|S_{b,c}(f)\|_{L^q_\eta}^p &\lesssim \left(\int_{\mathbb{D}} \left(\int_{\mathbb{D}} |f(\zeta)|^p h(\zeta)^p \frac{dA(\zeta)}{|1 - z\bar{\zeta}|^c} \right)^{\frac{q}{p}} \frac{\eta(z)}{\widehat{\eta}(z)^{\frac{1}{p'}}} dA(z) \right)^{\frac{p}{q}} \\ &\leq \int_{\mathbb{D}} |f(\zeta)|^p \widetilde{\sigma}(\zeta) I_3(\zeta) dA(\zeta), \end{aligned}$$

where

$$I_3(\zeta) = \frac{h(\zeta)^p}{\tilde{\sigma}(\zeta)} \left(\int_{\mathbb{D}} \frac{\eta(z) dA(z)}{|1 - z\bar{\zeta}|^{\frac{cq}{p}-1} \widehat{\eta}(z)^{\frac{1}{p'}}} \right)^{\frac{p}{q}} \asymp \frac{h(\zeta)^p}{\tilde{\sigma}(\zeta)} \left(\int_0^1 \frac{\eta(r)}{(1-r|\zeta|)^{\frac{cq}{p}-1} \widehat{\eta}(r)^{\frac{1}{p'}}} dr \right)^{\frac{p}{q}}.$$

Since

$$\int_0^{|\zeta|} \frac{\eta(r)}{(1-r|\zeta|)^{\frac{cq}{p}-1} \widehat{\eta}(r)^{\frac{1}{p'}}} dr \leq \int_0^{|\zeta|} \frac{\eta(r)}{(1-r)^{\frac{cq}{p}-1} \widehat{\eta}(r)^{\frac{1}{p'}}} dr \lesssim \frac{\widehat{\eta}(\zeta)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{cq}{p}-1}}, \quad \zeta \in \mathbb{D},$$

by the hypothesis (ii), and

$$\int_{|\zeta|}^1 \frac{\eta(r)}{(1-r|\zeta|)^{\frac{cq}{p}-1} \widehat{\eta}(r)^{\frac{1}{p'}}} dr \leq \frac{1}{(1-|\zeta|)^{\frac{cq}{p}-1}} \int_{|\zeta|}^1 \frac{\eta(r)}{\widehat{\eta}(r)^{\frac{1}{p'}}} dr \asymp \frac{\widehat{\eta}(\zeta)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{cq}{p}-1}}, \quad \zeta \in \mathbb{D},$$

we deduce

$$I_3(\zeta) \lesssim (1-|\zeta|)^{2+b-c+\frac{1}{q}-\frac{1}{p}} \frac{\widehat{\eta}(\zeta)^{\frac{1}{q}}}{\widehat{\sigma}(\zeta)^{\frac{1}{p}}} \lesssim 1, \quad \zeta \in \mathbb{D},$$

by the assumption (3). It follows that $\|S_{b,c}(f)\|_{L_{\eta}^q} \lesssim \|f\|_{A_{\sigma}^p}$. This finishes the proof because $\|f\|_{A_{\sigma}^p} \asymp \|f\|_{A_{\sigma}^p}$ for all $f \in \mathcal{H}(\mathbb{D})$ by Lemma 5 provided $\sigma \in \mathcal{D}$. \square

5 Proof of Theorem 1

In order to prove the sufficiency part of Theorem 1 we shall use the next result which follows from the argument used in the proof of [12, Lemma 4.5].

Lemma 13 *Let $1 < q < \infty$ and v, ω weights such that $P_{\omega} : L_v^q \rightarrow L_v^q$ is bounded. Then*

$$\|H_f^v(g)\|_{L_v^q}^q \leq (1 + \|P_{\omega}\|_{L_v^q \rightarrow L_v^q}) \|H_f^{\omega}(g)\|_{L_v^q}^q, \quad f \in L_v^q, \quad g \in H^{\infty}.$$

Proposition 14 *Let $1 < p \leq q < \infty$, $v \in B_q$ a radial weight and $\omega \in \mathcal{D}$. If $f \in \text{BO}(\Delta)$, then $H_f^v : A_{\omega}^p \rightarrow L_v^q$ is bounded.*

Proof By [5] there exists a constant $s_0 = s_0(v) > -1$ such that $P_s : L_v^q \rightarrow L_v^q$ is bounded for each $s > s_0$. Let $0 < \tau < \min\{q\alpha(\omega)/p, \alpha(v)\}$, where $\alpha(v)$ and $\alpha(\omega)$ are those from Lemma B. Then Lemmas 9 and 13 yield

$$\begin{aligned} \|H_f^v(g)\|_{L_v^q}^q &\lesssim \|H_f^s(g)\|_{L_v^q}^q \leq \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{|f(z) - f(\zeta)| |g(\zeta)|}{|1 - \bar{z}\zeta|^{2+s}} (1-|\zeta|^2)^s dA(\zeta) \right)^q v(z) dA(z) \\ &\lesssim \int_{\mathbb{D}} \left(\int_{\mathbb{D}} |g(\zeta)| \frac{(\beta(z, \zeta) + 1) \Gamma_{\tau}(z, \zeta)}{|1 - \bar{z}\zeta|^{2+s}} (1-|\zeta|^2)^s dA(\zeta) \right)^q v(z) dA(z), \quad g \in H^{\infty}. \end{aligned}$$

Let $s > \max\{s_0, 2(\beta(\omega) + \beta(v) + 2\alpha(v))\}$, $\delta < \min\{\frac{\tau}{q}, \frac{\alpha(v)}{q}\}$ and $K > 1$ to be fixed later. Then applying (3.8), we get

$$\begin{aligned} \|H_f^v(g)\|_{L_v^q}^q &\lesssim \sum_{j=1}^5 \int_{\mathbb{D}} \left(\int_{\Omega_j(z)} |g(\zeta)| \frac{\Gamma_{\tau}(z, \zeta) dA(\zeta)}{|1 - \bar{z}\zeta|^{2+s-2\delta} (1-|\zeta|^2)^{\delta-s}} \right)^q \frac{v(z)}{(1-|z|)^{q\delta}} dA(z) \\ &= \sum_{j=1}^5 I_j(g), \end{aligned} \tag{5.1}$$

where

$$\begin{aligned}\Omega_1(z) &= \left\{ \zeta \in \mathbb{D} : \frac{1}{|1 - \bar{z}\zeta|^2} \leq \frac{2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \right\} \cap D(0, |z|), \\ \Omega_2(z) &= \left\{ \zeta \in \mathbb{D} : \frac{1}{|1 - \bar{z}\zeta|^2} \leq \frac{2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \right\} \cap (\mathbb{D} \setminus D(0, |z|)), \\ \Omega_3(z, K) &= \left\{ \zeta \in \mathbb{D} : \frac{1 - |\zeta|}{K} \geq \frac{2|1 - \bar{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \right\}, \\ \Omega_4(z, K) &= \left\{ \zeta \in \mathbb{D} : \frac{1 - |\zeta|}{K} < \frac{2|1 - \bar{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} < 1 \right\} \cap D(0, |z|), \\ \Omega_5(z, K) &= \left\{ \zeta \in \mathbb{D} : \frac{1 - |\zeta|}{K} < \frac{2|1 - \bar{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} < 1 \right\} \cap (\mathbb{D} \setminus D(0, |z|)).\end{aligned}$$

The quantities $I_j(g)$, $j = 1, \dots, 5$, will be estimated separately.

Case $I_1(g)$ By using the definition of $\Omega_1(z)$, and the fact that $\frac{\widehat{v}(x)}{(1-x)^\tau}$ is essentially decreasing on $[0, 1)$ we deduce

$$\Gamma_\tau(z, \zeta) \lesssim \frac{\left(\frac{|1 - \bar{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \right)^{\frac{1}{p} - \frac{1}{q}}}{\min \left\{ \frac{\widehat{v}(z)}{(1 - |z|)^\tau}, \frac{\widehat{v}(\zeta)}{(1 - |\zeta|)^\tau} \right\}^{\frac{1}{q}}} \lesssim \left(\frac{|1 - \bar{z}\zeta|^2}{1 - |\zeta|^2} \right)^{\frac{1}{p} - \frac{1}{q}} \left(\frac{(1 - |z|)^\tau}{\widehat{v}(z)} \right)^{\frac{1}{q}}, \quad z \in \mathbb{D}, \quad \zeta \in \Omega_1(z).$$

Then the estimate

$$M_1(r, f) \leq M_p(r, f) \lesssim \|f\|_{A_\omega^p} \widehat{\omega}(r)^{-\frac{1}{p}}, \quad 0 \leq r < 1, \quad f \in \mathcal{H}(\mathbb{D}), \quad (5.2)$$

and Lemma 3(ii) yield

$$\begin{aligned}I_1(g) &\lesssim \int_{\mathbb{D}} \left(\int_{\Omega_1(z)} |g(\zeta)| \frac{(1 - |\zeta|)^{s - \delta - \frac{1}{p} + \frac{1}{q}}}{|1 - \bar{z}\zeta|^{2 + s - 2\delta - 2(\frac{1}{p} - \frac{1}{q})}} dA(\zeta) \right)^q \frac{v(z)(1 - |z|)^{\tau - \delta q}}{\widehat{v}(z)} dA(z) \\ &\lesssim \left(\int_{\mathbb{D}} |g(\zeta)|(1 - |\zeta|)^{\frac{s}{2} - 1} dA(\zeta) \right)^q \int_{\mathbb{D}} \frac{v(z)(1 - |z|)^{\tau - \delta q}}{\widehat{v}(z)} dA(z) \\ &\lesssim \|g\|_{A_\omega^p}^q \left(\int_0^1 \frac{(1 - t)^{\frac{s}{2} - 1}}{\widehat{\omega}(t)^{\frac{1}{p}}} dt \right)^q \lesssim \|g\|_{A_\omega^p}^q \left(\int_0^1 (1 - t)^{\frac{s}{2} - 1 - \frac{\beta(\omega)}{p}} dt \right)^q \lesssim \|g\|_{A_\omega^p}^q, \quad g \in H^\infty.\end{aligned}$$

Case $I_2(g)$ The definition of $\Omega_2(z)$ and the fact that $\frac{\widehat{v}(x)}{(1-x)^\tau}$ is essentially decreasing imply

$$\Gamma_\tau(z, \zeta) \lesssim \frac{\left(\frac{|1 - \bar{z}\zeta|^2}{\max\{1 - |z|^2, 1 - |\zeta|^2\}} \right)^{\frac{1}{p} - \frac{1}{q}}}{\min \left\{ \frac{\widehat{v}(z)}{(1 - |z|)^\tau}, \frac{\widehat{v}(\zeta)}{(1 - |\zeta|)^\tau} \right\}^{\frac{1}{q}}} \lesssim \left(\frac{|1 - \bar{z}\zeta|^2}{1 - |\zeta|^2} \right)^{\frac{1}{p} - \frac{1}{q}} \left(\frac{(1 - |\zeta|)^\tau}{\widehat{v}(\zeta)} \right)^{\frac{1}{q}}, \quad z \in \mathbb{D}, \quad \zeta \in \Omega_2(z).$$

Therefore (5.2) and Lemmas A and B yield

$$\begin{aligned}
 I_2(g) &\lesssim \int_{\mathbb{D}} \left(\int_{\Omega_2(z)} |g(\zeta)| \frac{(1-|\zeta|)^{s-\delta-\frac{1}{p}+\frac{1+\tau}{q}}}{\widehat{v}(\zeta)^{\frac{1}{q}} |1-\bar{z}\zeta|^{2+s-2\delta-2(\frac{1}{p}-\frac{1}{q})}} dA(\zeta) \right)^q (1-|z|)^{-\delta q} v(z) dA(z) \\
 &\lesssim \left(\int_{\mathbb{D}} |g(\zeta)| \frac{(1-|\zeta|)^{\frac{s}{2}-1+\frac{\tau}{q}}}{\widehat{v}(\zeta)^{\frac{1}{q}}} dA(\zeta) \right)^q \int_0^1 (1-r)^{-\delta q} v(r) dr \\
 &\lesssim \|g\|_{A_{\omega}^p}^q \left(\int_0^1 \frac{(1-r)^{\frac{s}{2}-1+\frac{\tau}{q}}}{\widehat{\omega}(r)^{\frac{1}{p}} \widehat{v}(\zeta)^{\frac{1}{q}}} dA(\zeta) \right)^q \left(\widehat{v}(0) + \int_0^1 \frac{\widehat{v}(t)}{(1-t)^{1+q\delta}} dt \right) \\
 &\lesssim \|g\|_{A_{\omega}^p}^q \left(\int_0^1 (1-t)^{\frac{s}{2}-1-\frac{\beta(\omega)}{p}-\frac{\beta(v)-\tau}{q}} dt \right)^q \lesssim \|g\|_{A_{\omega}^p}^q, \quad g \in H^{\infty}.
 \end{aligned}$$

Case $I_3(g)$ To deal with $I_3(g)$, note first that now $2K|1-\bar{z}\zeta|^2 \leq (1-|\zeta|) \max\{1-|z|^2, 1-|\zeta|^2\} \leq 2(\max\{1-|z|, 1-|\zeta|\})^2$ for all $\zeta \in \Omega_3(z, K)$. Hence $\zeta \in \Delta(z, R)$ for some $R = R(K) \in (0, \infty)$ if $K \geq 1$ is sufficiently large. Fix such a K , and note that then $\widehat{v}(\zeta) \asymp \widehat{v}(z)$ for all $\zeta \in \Omega(z, K)$ by Lemma A(ii). By using this and the fact that $\frac{\widehat{\omega}(x)^{\frac{p}{p-1}}}{(1-x)^{\frac{p}{q}}}$ is essentially decreasing on $[0, 1)$ we deduce

$$\begin{aligned}
 \Gamma_{\tau}(z, \zeta) &\lesssim \left(\frac{|1-\bar{z}\zeta|^2}{\max\{1-|z|^2, 1-|\zeta|^2\}} \right)^{\frac{1}{p}-\frac{1}{q}} \frac{\widehat{\omega}(\zeta)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{\tau}{q}}} \min \left\{ \frac{\widehat{v}(z)}{(1-|z|)^{\tau}}, \frac{\widehat{v}(\zeta)}{(1-|\zeta|)^{\tau}} \right\}^{-\frac{1}{q}} \\
 &\asymp \frac{(1-|\zeta|)^{\frac{1}{p}-\frac{1}{q}} \widehat{\omega}(\zeta)^{\frac{1}{p}}}{\widehat{v}(\zeta)^{\frac{1}{q}}}, \quad z \in \mathbb{D}, \quad \zeta \in \Omega_3(z, K),
 \end{aligned}$$

and it follows that

$$\begin{aligned}
 I_3(g) &\lesssim \int_{\mathbb{D}} \left(\int_{\Delta(z, R)} \left(|g(\zeta)| \widehat{\omega}(\zeta)^{\frac{1}{p}} \right) \frac{(1-|\zeta|^2)^{s-\delta+\frac{1}{p}-\frac{2}{q}}}{|1-\bar{z}\zeta|^{2+s-2\delta}} dA(\zeta) \right)^q \frac{v(z)(1-|z|)^{1-q\delta}}{\widehat{v}(z)} dA(z) \\
 &\asymp \int_{\mathbb{D}} \left(\int_{\Delta(z, R)} \left(|g(\zeta)| \widehat{\omega}(\zeta)^{\frac{1}{p}} \right) \frac{(1-|\zeta|^2)^{s+\frac{2}{p}-\frac{2}{q}}}{|1-\bar{z}\zeta|^{2+s}} dA(\zeta) \right)^q \frac{v(z)(1-|z|)}{\widehat{v}(z)} dA(z) \\
 &\asymp \int_{\mathbb{D}} \left(\int_{\Delta(z, R)} \left(|g(\zeta)| \widehat{\omega}(\zeta)^{\frac{1}{p}} \right) \frac{(1-|\zeta|^2)^s dA(\zeta)}{|1-\bar{z}\zeta|^{2+s-\frac{2}{p}+\frac{2}{q}}} \right)^q \frac{v(z)(1-|z|)}{\widehat{v}(z)} dA(z) \tag{5.3} \\
 &\leq \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \left(|g(\zeta)| \widehat{\omega}(\zeta)^{\frac{1}{p}} \right) \frac{(1-|\zeta|^2)^s dA(\zeta)}{|1-\bar{z}\zeta|^{2+s-\frac{2}{p}+\frac{2}{q}}} \right)^q \frac{v(z)(1-|z|)}{\widehat{v}(z)} dA(z) \\
 &= \left\| S_{s, s+2(1-\frac{1}{p}+\frac{1}{q})} \left(|g| \widehat{\omega}^{\frac{1}{p}} \right) \right\|_{L_{\eta}^q}^q = \left\| S_{b, c} \left(|g| \widehat{\omega}^{\frac{1}{p}} \right) \right\|_{L_{\eta}^q}^q, \quad g \in H^{\infty},
 \end{aligned}$$

where $\eta(z) = \frac{v(z)(1-|z|)}{\widehat{v}(z)}$ for all $z \in \mathbb{D}$. To apply Lemma 12 with $\sigma \equiv 1$, we must check that its hypotheses are satisfied. To do this, first observe that $\eta \in \mathcal{D}$ and $\widehat{\eta}(r) \asymp (1-r)$ for all

$0 \leq r < 1$ by Lemma 4. Hence

$$\int_r^1 \frac{(1-t)^{c-2}}{\widehat{\eta}(t)^{\frac{1}{q}}} dt \asymp \int_r^1 (1-t)^{s-\frac{2}{p}+\frac{1}{q}} dt \asymp (1-r)^{1+s-\frac{2}{p}+\frac{1}{q}} \asymp \frac{(1-r)^{c-1}}{\widehat{\eta}(r)^{\frac{1}{q}}}, \quad 0 \leq r < 1,$$

and, by Lemma 3(iii),

$$\begin{aligned} \int_0^r \frac{\eta(t)}{(1-t)^{\frac{cq}{p}-1} \widehat{\eta}(t)^{\frac{1}{p'}}} dt &\asymp \int_0^r \frac{v(t)}{\widehat{v}(t)(1-t)^{\frac{q}{p}(s+2(1-\frac{1}{p}+\frac{1}{q})) - 1 - \frac{1}{p}}} dt \\ &\lesssim \frac{1}{(1-r)^{\frac{q}{p}(s+2(1-\frac{1}{p}+\frac{1}{q})) - 1 - \frac{1}{p}}} \asymp \frac{\widehat{\eta}(r)^{\frac{1}{p}}}{(1-r)^{\frac{cq}{p}-1}}, \quad 0 \leq r < 1, \end{aligned}$$

so the hypotheses of Lemma 12 are satisfied. Moreover,

$$(1-r)^{2+b-c+\frac{1}{q}-\frac{1}{p}} \frac{\widehat{\eta}(r)^{\frac{1}{q}}}{\widehat{\sigma}(r)^{\frac{1}{p}}} \asymp 1, \quad 0 \leq r < 1,$$

and consequently (5.3) and Lemmas 12 and 5 yield $I_3(g) \lesssim \|g\|_{A_{\omega}^p}^q \asymp \|g\|_{A_{\omega}^p}^q$ for all $g \in H^\infty$.

Case $I_4(g)$ By using the definition of $\Omega_4(z, K)$, Lemma A(ii) and the fact that $\frac{\widehat{v}(x)}{(1-x)^\tau}$ is essentially decreasing on $[0, 1)$, we deduce

$$\begin{aligned} \Gamma_\tau(z, \zeta) &\lesssim \frac{\left(\frac{|1-\bar{z}\zeta|^2}{\max\{|1-|z|^2|, |1-|\zeta|^2|\}}\right)^{\frac{1}{p}-\frac{1}{q}} \widehat{\omega}\left(1 - \frac{2|1-\bar{z}\zeta|^2}{\max\{|1-|z|^2|, |1-|\zeta|^2|\}}\right)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{\tau}{q}} \min\left\{\frac{\widehat{v}(z)}{(1-|z|)^\tau}, \frac{\widehat{v}(\zeta)}{(1-|\zeta|)^\tau}\right\}^{\frac{1}{q}}} \\ &\lesssim \frac{\left(\frac{|1-\bar{z}\zeta|^2}{1-|\zeta|^2}\right)^{\frac{1}{p}-\frac{1}{q}} \widehat{\omega}\left(1 - \frac{K2|1-\bar{z}\zeta|^2}{\max\{|1-|z|^2|, |1-|\zeta|^2|\}}\right)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{\tau}{q}} \min\left\{\frac{\widehat{v}(z)}{(1-|z|)^\tau}, \frac{\widehat{v}(\zeta)}{(1-|\zeta|)^\tau}\right\}^{\frac{1}{q}}} \lesssim \frac{\left(\frac{|1-\bar{z}\zeta|^2}{1-|\zeta|^2}\right)^{\frac{1}{p}-\frac{1}{q}} \left(\frac{|1-\bar{z}\zeta|}{1-|\zeta|}\right)^{\frac{2\beta(\omega)}{p}} \widehat{\omega}(\zeta)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{\tau}{q}} \min\left\{\frac{\widehat{v}(z)}{(1-|z|)^\tau}, \frac{\widehat{v}(\zeta)}{(1-|\zeta|)^\tau}\right\}^{\frac{1}{q}}} \\ &\lesssim \frac{|1-\bar{z}\zeta|^{\frac{2\beta(\omega)}{p}+\frac{2}{p}-\frac{2}{q}}}{(1-|\zeta|)^{\frac{2\beta(\omega)}{p}+\frac{1}{p}-\frac{1}{q}}} \frac{\widehat{\omega}(\zeta)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{\tau}{q}}} \left(\frac{(1-|z|)^\tau}{\widehat{v}(z)}\right)^{\frac{1}{q}}, \quad z \in \mathbb{D}, \quad \zeta \in \Omega_4(z, K). \end{aligned}$$

Therefore

$$\begin{aligned} I_4(g) &\lesssim \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \left(|g(\zeta)| \widehat{\omega}(\zeta)^{\frac{1}{p}} \right) \frac{(1-|\zeta|)^{s-\delta-\frac{2\beta(\omega)}{p}+\frac{1}{q}-\frac{\tau}{q}}}{|1-\bar{z}\zeta|^{2+s-2\delta-\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q}}} dA(\zeta) \right)^q \frac{v(z)(1-|z|)^{\tau-\delta q}}{\widehat{v}(z)} dA(z) \\ &= \|S_{b,c} \left(|g| \widehat{\omega}^{\frac{1}{p}} \right)\|_{L_\eta^q}^q, \quad g \in H^\infty, \end{aligned} \quad (5.4)$$

where $b = s - \delta - \frac{2\beta(\omega)}{p} + \frac{1}{q} - \frac{\tau}{q}$, $c = 2 + s - 2\delta - \frac{2\beta(\omega)}{p} - \frac{2}{p} + \frac{2}{q}$ and $\eta(z) = \frac{v(z)(1-|z|)^{\tau-\delta q}}{\widehat{v}(z)}$ for all $z \in \mathbb{D}$. We will appeal to Lemma 12 with $\sigma \equiv 1$. First observe that $\eta \in \mathcal{D}$ and $\widehat{\eta}(r) \asymp (1-r)^{\tau-\delta q}$ for all $0 \leq r < 1$ by Lemma 4. Hence

$$\begin{aligned} \int_r^1 \frac{(1-t)^{c-2}}{\widehat{\eta}(t)^{\frac{1}{q}}} dt &\asymp \int_r^1 (1-t)^{s-\delta-\frac{2\beta(\omega)}{p}-\frac{\tau}{q}-\frac{2}{p}+\frac{2}{q}} dt \asymp (1-r)^{1+s-\delta-\frac{2\beta(\omega)}{p}-\frac{\tau}{q}-\frac{2}{p}+\frac{2}{q}} \\ &\asymp \frac{(1-r)^{c-1}}{\widehat{\eta}(r)^{\frac{1}{q}}}, \quad 0 \leq r < 1, \end{aligned}$$

and, by Lemma 3(iii),

$$\begin{aligned} \int_0^r \frac{\eta(t)}{(1-t)^{\frac{cq}{p}-1} \widehat{\eta}(t)^{\frac{1}{p'}}} dt &\asymp \int_0^r \frac{v(t)}{\widehat{v}(t)(1-t)^{\frac{q}{p}(2+s-2\delta-\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q})-1-\frac{\tau-q\delta}{p}}} dt \\ &\lesssim \frac{1}{(1-r)^{\frac{q}{p}(2+s-2\delta-\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q})-1-\frac{\tau-q\delta}{p}}} \asymp \frac{\widehat{\eta}(r)^{\frac{1}{p}}}{(1-r)^{\frac{cq}{p}-1}}, \quad 0 \leq r < 1, \end{aligned}$$

so the hypotheses of Lemma 12 are satisfied. Moreover,

$$(1-r)^{2+b-c+\frac{1}{q}-\frac{1}{p}} \frac{\widehat{\eta}(r)^{\frac{1}{q}}}{\widehat{\sigma}(r)^{\frac{1}{p}}} \asymp 1, \quad 0 \leq r < 1,$$

and hence (5.4) and Lemmas 12 and 5 imply $I_4(g) \lesssim \|g\|_{A_{\omega}^p}^q \asymp \|g\|_{A_{\omega}^p}^q$ for all $g \in H^\infty$.

Case $I_5(g)$ By using the definition of $\Omega_5(z, K)$, Lemma A(ii) and the fact that $\frac{\widehat{v}(x)}{(1-x)^\tau}$ is essentially decreasing on $[0, 1)$ we deduce

$$\begin{aligned} \Gamma_\tau(z, \zeta) &\lesssim \frac{\left(\frac{|1-\bar{z}\zeta|^2}{\max\{|1-|z|^2, 1-|\zeta|^2\}} \right)^{\frac{1}{p}-\frac{1}{q}} \widehat{\omega} \left(1 - \frac{2|1-\bar{z}\zeta|^2}{\max\{|1-|z|^2, 1-|\zeta|^2\}} \right)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{\tau}{q}} \min \left\{ \frac{\widehat{v}(z)}{(1-|z|)^\tau}, \frac{\widehat{v}(\zeta)}{(1-|\zeta|)^\tau} \right\}^{\frac{1}{q}}} \\ &\lesssim \frac{\left(\frac{|1-\bar{z}\zeta|^2}{1-|\zeta|^2} \right)^{\frac{1}{p}-\frac{1}{q}} \widehat{\omega} \left(1 - \frac{K2|1-\bar{z}\zeta|^2}{\max\{|1-|z|^2, 1-|\zeta|^2\}} \right)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{\tau}{q}} \min \left\{ \frac{\widehat{v}(z)}{(1-|z|)^\tau}, \frac{\widehat{v}(\zeta)}{(1-|\zeta|)^\tau} \right\}^{\frac{1}{q}}} \lesssim \frac{\left(\frac{|1-\bar{z}\zeta|^2}{1-|\zeta|^2} \right)^{\frac{1}{p}-\frac{1}{q}} \left(\frac{|1-\bar{z}\zeta|}{1-|\zeta|^2} \right)^{\frac{2\beta(\omega)}{p}} \widehat{\omega}(\zeta)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{\tau}{q}} \min \left\{ \frac{\widehat{v}(z)}{(1-|z|)^\tau}, \frac{\widehat{v}(\zeta)}{(1-|\zeta|)^\tau} \right\}^{\frac{1}{q}}} \\ &\lesssim \left(\frac{|1-\bar{z}\zeta|^2}{1-|\zeta|^2} \right)^{\frac{1}{p}-\frac{1}{q}} \left(\frac{|1-\bar{z}\zeta|}{1-|\zeta|} \right)^{\frac{2\beta(\omega)}{p}} \frac{\widehat{\omega}(\zeta)^{\frac{1}{p}}}{\widehat{v}(\zeta)^{\frac{1}{q}}}, \quad z \in \mathbb{D}, \quad \zeta \in \Omega_5(z, K). \end{aligned}$$

Therefore Lemma A(ii) yields

$$\begin{aligned} I_5(g) &\lesssim \int_{\mathbb{D}} \left(\int_{\Omega_5(z, K)} \left(|g(\zeta)| \widehat{\omega}(\zeta)^{\frac{1}{p}} \right) \frac{(1-|\zeta|)^{s-\delta-\frac{2\beta(\omega)}{p}+\frac{1}{q}}}{\widehat{v}(\zeta)^{\frac{1}{q}} |1-\bar{z}\zeta|^{2+s-2\delta-\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q}}} dA(\zeta) \right)^q \frac{v(z) dA(z)}{(1-|z|)^{q\delta}} \\ &\lesssim \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \left(|g(\zeta)| \widehat{\omega}(\zeta)^{\frac{1}{p}} \right) \frac{(1-|\zeta|)^{s-\delta-\frac{2\beta(\omega)}{p}+\frac{1}{q}-\frac{\beta(v)}{q}}}{|1-\bar{z}\zeta|^{2+s-2\delta-\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q}}} dA(\zeta) \right)^q \frac{v(z) dA(z)}{(1-|z|)^{q\delta-\beta(v)} \widehat{v}(z)^{\frac{1}{q}}} \quad (5.5) \\ &= \left\| S_{b,c} \left(|g| \widehat{\omega}^{\frac{1}{p}} \right) \right\|_{L_{\eta}^q}^q, \quad g \in H^\infty, \end{aligned}$$

where $b = s - \delta - \frac{2\beta(\omega)}{p} + \frac{1}{q} - \frac{\beta(v)}{q}$, $c = 2 + s - 2\delta - \frac{2\beta(\omega)}{p} - \frac{2}{p} + \frac{2}{q}$ and $\eta(z) = \frac{v(z)(1-|z|)^{\beta(v)-\delta q}}{\widehat{v}(z)}$ for all $z \in \mathbb{D}$. Again we will appeal to Lemma 12 with $\sigma \equiv 1$. First observe that $\eta \in \mathcal{D}$ and $\widehat{\eta}(r) \asymp (1-r)^{\beta(v)-\delta q}$ for all $0 \leq r < 1$ by Lemma 4. Hence

$$\begin{aligned} \int_r^1 \frac{(1-t)^{c-2}}{\widehat{\eta}(t)^{\frac{1}{q}}} dt &\asymp \int_r^1 (1-t)^{s-\delta+\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q}-\frac{\beta(v)}{q}} dt \asymp (1-r)^{1+s-\delta+\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q}-\frac{\beta(v)}{q}} \\ &\asymp \frac{(1-r)^{c-1}}{\widehat{\eta}(r)^{\frac{1}{q}}}, \quad 0 \leq r < 1, \end{aligned}$$

and, by Lemma 3(iii),

$$\begin{aligned} \int_0^r \frac{\eta(t)}{(1-t)^{\frac{cq}{p}-1} \widehat{\eta}(t)^{1/p'}} dt &\asymp \int_0^r \frac{v(t)}{\widehat{v}(t)(1-t)^{\frac{q}{p} \left(2+s-2\delta - \frac{2\beta(\omega)}{p} - \frac{2}{p} + \frac{2}{q}\right) - 1 - \frac{\beta(v)-q\delta}{p}}} dt \\ &\lesssim \frac{1}{(1-r)^{\frac{q}{p} \left(2+s-2\delta - \frac{2\beta(\omega)}{p} - \frac{2}{p} + \frac{2}{q}\right) - 1 - \frac{\beta(v)-q\delta}{p}}} \asymp \frac{\widehat{\eta}(r)^{\frac{1}{p}}}{(1-r)^{\frac{cq}{p}-1}}, \quad 0 \leq r < 1, \end{aligned}$$

so the hypotheses of Lemma 12 are satisfied. Moreover,

$$(1-r)^{2+b-c+\frac{1}{q}-\frac{1}{p}} \frac{\widehat{\eta}(r)^{\frac{1}{q}}}{\widehat{\sigma}(r)^{\frac{1}{p}}} \asymp 1, \quad 0 \leq r < 1,$$

and hence (5.5) together with Lemmas 5 and 12 imply $I_5(g) \lesssim \|g\|_{A_\omega^p}^q \asymp \|g\|_{A_\omega^p}^q$ for all $g \in H^\infty$. This finishes the proof of the proposition. \square

In order to prove the necessity part of Theorem 1 some definitions are needed. For $\eta > -1$ and a radial weight ω , let $b_{z,\omega}^\eta = B_z^\eta / \|B_z^\eta\|_{A_\omega^p}$ for $z \in \mathbb{D}$, where $B_z^\eta(\zeta) = (1 - \bar{z}\zeta)^{-(2+\eta)}$. For each $f \in L_v^1$, define

$$g_{z,\omega}^\eta(\zeta) = \frac{P_v(\bar{f} b_{z,\omega}^\eta)(\zeta)}{b_{z,\omega}^\eta(\zeta)}, \quad \zeta \in \mathbb{D},$$

and note that $g_{z,\omega}^\eta$ is a well-defined analytic function in \mathbb{D} because the standard Bergman kernel $b_{z,\omega}^\eta$ has no zeros. If ν, ω are weights, $\eta > -1$ and $0 < p, q < \infty$, let us consider the global mean oscillation

$$\|f b_{z,\omega}^\eta - \overline{g_{z,\omega}^\eta(z)} b_{z,\omega}^\eta\|_{L_v^q}, \quad z \in \mathbb{D}.$$

Proposition 15 *Let $1 < p \leq q < \infty$, $f \in L_v^q$, $\omega \in \widehat{\mathcal{D}}$, $\nu \in B_q$ a radial weight and $\gamma(z) = \gamma_{\omega,\nu,p,q}(z) = \frac{\widehat{\nu}(z)^{\frac{1}{q}}(1-|z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}}(1-|z|)^{\frac{1}{p}}}$. If $H_f^\nu, H_{\bar{f}}^\nu : A_\omega^p \rightarrow L_v^q$ are bounded, then there exists $\eta_0 = \eta_0(\nu, \omega) > -1$ such that*

$$\sup_{z \in \mathbb{D}} \|f b_{z,\omega}^\eta - \overline{g_{z,\omega}^\eta(z)} b_{z,\omega}^\eta\|_{L_v^q} \leq \|H_f^\nu\|_{A_\omega^p \rightarrow L_v^q} + \|P_\eta\|_{L_v^q \rightarrow L_v^q} \left(\|H_f^\nu\|_{A_\omega^p \rightarrow L_v^q} + \|H_{\bar{f}}^\nu\|_{A_\omega^p \rightarrow L_v^q} \right).$$

for each $\eta \geq \eta_0$. Moreover, there exists $r_0 = r_0(\nu) > 0$ such that for each fixed $r \geq r_0$ and $\eta \geq \eta_0$,

$$\sup_{z \in \mathbb{D}} \|f b_{z,\omega}^\eta - \overline{g_{z,\omega}^\eta(z)} b_{z,\omega}^\eta\|_{L_v^q} \gtrsim \sup_{z \in \mathbb{D}} \gamma(z) \left(\frac{1}{v(\Delta(z, r))} \int_{\Delta(z, r)} |f(\zeta) - \overline{g_{z,\omega}^\eta(z)}|^q v(\zeta) dA(\zeta) \right)^{\frac{1}{q}}.$$

Proof The definition of the Hankel operator along with triangle inequality gives

$$\begin{aligned} \|f b_{z,\omega}^\eta - \overline{g_{z,\omega}^\eta(z)} b_{z,\omega}^\eta\|_{L_v^q} &\leq \|H_f^\nu(b_{z,\omega}^\eta)\|_{L_v^q} + \|P_\nu(f b_{z,\omega}^\eta) - \overline{g_{z,\omega}^\eta(z)} b_{z,\omega}^\eta\|_{L_v^q} \\ &\leq \|H_f^\nu\|_{A_\omega^p \rightarrow L_v^q} \|b_{z,\omega}^\eta\|_{A_\omega^p} + \|P_\nu(f b_{z,\omega}^\eta) - \overline{g_{z,\omega}^\eta(z)} b_{z,\omega}^\eta\|_{L_v^q} \\ &= \|H_f^\nu\|_{A_\omega^p \rightarrow L_v^q} + \|P_\nu(f b_{z,\omega}^\eta) - \overline{g_{z,\omega}^\eta(z)} b_{z,\omega}^\eta\|_{L_v^q}. \end{aligned}$$

If $g \in A_\eta^1$, then the reproducing formula for the standard weighted Bergman projection yields $\overline{g(z)} b_{z,\omega}^\eta = P_\eta(\overline{g} b_{z,\omega}^\eta)$. Since $\nu \in B_q$ is radial and $f \in L_v^q$, we have $\nu \in \mathcal{D}$ and $P_\nu(f b_{z,\omega}^\eta) \in A_\nu^q$

by Proposition 6. Therefore $g_z^\eta \in A_v^q$ for all $z \in \mathbb{D}$. Moreover, $A_v^q \subset A_\eta^q \subset A_\eta^1$ if $\eta > \frac{\beta(v)}{q} - 1$ by Lemma A(ii). It follows that

$$\begin{aligned} \|P_v(fb_{z,\omega}^\eta) - \overline{g_{z,\omega}^\eta(z)}b_{z,\omega}^\eta\|_{L_v^q} &= \|P_v(fb_{z,\omega}^\eta) - P_\eta(\overline{g_{z,\omega}^\eta}b_{z,\omega}^\eta)\|_{L_v^q} \\ &= \|P_\eta(P_v(fb_{z,\omega}^\eta) - \overline{g_{z,\omega}^\eta}b_{z,\omega}^\eta)\|_{L_v^q}, \quad z \in \mathbb{D}. \end{aligned}$$

By [5], there exists $\eta_1 = \eta_1(v) > \frac{\beta(v)}{q} - 1$ such that $P_\eta : L_v^q \rightarrow L_v^q$ is bounded if $\eta \geq \eta_1$. Therefore

$$\|P_v(fb_{z,\omega}^\eta) - \overline{g_{z,\omega}^\eta(z)}b_{z,\omega}^\eta\|_{L_v^q} \leq \|P_\eta\|_{L_v^q \rightarrow L_v^q} \|P_v(fb_{z,\omega}^\eta) - \overline{g_{z,\omega}^\eta}b_{z,\omega}^\eta\|_{L_v^q}, \quad z \in \mathbb{D}, \quad \eta \geq \eta_1.$$

The triangle inequality yields

$$\begin{aligned} \|P_v(fb_{z,\omega}^\eta) - \overline{g_{z,\omega}^\eta}b_{z,\omega}^\eta\|_{L_v^q} &\leq \|fb_{z,\omega}^\eta - P_v(fb_{z,\omega}^\eta)\|_{L_v^q} + \|fb_{z,\omega}^\eta - \overline{g_{z,\omega}^\eta}b_{z,\omega}^\eta\|_{L_v^q} \\ &= \|H_f^v(b_{z,\omega}^\eta)\|_{L_v^q} + \|\bar{f}b_{z,\omega}^\eta - g_{z,\omega}^\eta b_{z,\omega}^\eta\|_{L_v^q} \\ &\leq \|H_f^v\|_{A_\omega^p \rightarrow L_v^q} \|b_{z,\omega}^\eta\|_{A_\omega^p} + \|\bar{f}b_{z,\omega}^\eta - P_v(\bar{f}b_{z,\omega}^\eta)\|_{L_v^q} \\ &= \|H_f^v\|_{A_\omega^p \rightarrow L_v^q} + \|H_{\bar{f}}^v(b_{z,\omega}^\eta)\|_{L_v^q} \leq \|H_f^v\|_{A_\omega^p \rightarrow L_v^q} + \|H_{\bar{f}}^v\|_{A_\omega^p \rightarrow L_v^q}. \end{aligned}$$

By combining the above estimates we deduce

$$\|fb_{z,\omega}^\eta - \overline{g_{z,\omega}^\eta(z)}b_{z,\omega}^\eta\|_{L_v^q} \leq \|H_f^v\|_{A_\omega^p \rightarrow L_v^q} + \|P_\eta\|_{L_v^q \rightarrow L_v^q} \left(\|H_f^v\|_{A_\omega^p \rightarrow L_v^q} + \|H_{\bar{f}}^v\|_{A_\omega^p \rightarrow L_v^q} \right),$$

for any $\eta \geq \eta_1(v)$.

To see the second one, first observe that [16, Corollary 2] and Lemma A(ii) give

$$\begin{aligned} \|B_z^\eta\|_{A_\omega^p}^p &\asymp \int_0^{|z|} \frac{\widehat{\omega}(t)}{(1-t)^{p(2+\eta)}} dt \lesssim \frac{\widehat{\omega}(|z|)}{(1-|z|)^{\beta(\omega)}} \int_0^{|z|} \frac{1}{(1-t)^{p(2+\eta)-\beta(\omega)}} dt \\ &\asymp \frac{\widehat{\omega}(z)}{(1-|z|)^{p(2+\eta)-1}}, \quad |z| \rightarrow 1^-, \end{aligned}$$

provided $\eta > \frac{\beta(\omega)+1}{p} - 2$. Moreover, by (3.6) there exists $r_0 = r_0(v) > 0$ such that $(1-|z|)\widehat{v}(z) \asymp v(\Delta(z, r_0))$ for any $r \geq r_0$. Hence, for each $r \geq r_0$ we have

$$\begin{aligned} \|fb_{z,\omega}^\eta - \overline{g_{z,\omega}^\eta(z)}b_{z,\omega}^\eta\|_{L_v^q}^q &\geq \int_{\Delta(z,r)} |f(\zeta) - \overline{g_{z,\omega}^\eta(z)}| |b_{z,\omega}^\eta(\zeta)|^q v(\zeta) dA(\zeta) \\ &\asymp \frac{1}{\|B_z^\eta\|_{A_\omega^p}^q (1-|z|)^{q(2+\eta)}} \int_{\Delta(z,r)} |f(\zeta) - \overline{g_{z,\omega}^\eta(z)}| |b_{z,\omega}^\eta(\zeta)|^q v(\zeta) dA(\zeta) \\ &\asymp \frac{1}{\widehat{\omega}(z)^{\frac{q}{p}} (1-|z|)^{\frac{q}{p}}} \int_{\Delta(z,r)} |f(\zeta) - \overline{g_{z,\omega}^\eta(z)}| |b_{z,\omega}^\eta(\zeta)|^q v(\zeta) dA(\zeta), \\ &\asymp \frac{\widehat{v}(z)(1-|z|)}{\widehat{\omega}(z)^{\frac{q}{p}} (1-|z|)^{\frac{q}{p}}} \frac{1}{v(\Delta(z,r))} \int_{\Delta(z,r)} |f(\zeta) - \overline{g_{z,\omega}^\eta(z)}| |b_{z,\omega}^\eta(\zeta)|^q v(\zeta) dA(\zeta). \end{aligned}$$

The second claim for $\eta_0 = \max\{\eta_1, \frac{\beta(\omega)+1}{p} - 2\}$ is now proved. \square

Proof of Theorem 1 If $H_f^v, H_{\bar{f}}^v : A_\omega^p \rightarrow L_v^q$ are bounded, then $f \in \text{BMO}(\Delta)$ by Proposition 15 and Theorem 11.

Conversely, let $f \in \text{BMO}(\Delta)$. Then f can be decomposed as $f = f_1 + f_2$, where $f_1 \in \text{BA}(\Delta)$ and $f_2 \in \text{BO}(\Delta)$, by Theorem 11(ii). Proposition 14 shows that $H_{f_2}^v, H_{\bar{f}_2}^v :$

$A_\omega^p \rightarrow L_v^q$ are bounded. Moreover, since $v \in B_q$ is radial, $v \in \mathcal{D}$ and $P_v : L_v^q \rightarrow L_v^q$ is bounded by Proposition 6. Therefore Lemma 10 yields

$$\|H_{f_1}^v(g)\|_{L_v^q}^q \leq \|f_1 g\|_{L_v^q}^q + \|P_v(f_1 g)\|_{L_v^q}^q \lesssim \|f_1 g\|_{L_v^q}^q \lesssim \|g\|_{A_\omega^p}^q \quad g \in H^\infty.$$

It follows that $H_f^v, H_{\bar{f}}^v : A_\omega^p \rightarrow L_v^q$ are bounded. \square

6 Anti-analytic symbols

Recall that the space $\mathcal{B}_{d\gamma}$ consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}_{d\gamma}} = \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|)\gamma(z) + |f(0)| < \infty,$$

where $\gamma(z) = \frac{\widehat{v}(z)^{\frac{1}{q}}(1-|z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}}(1-|z|)^{\frac{1}{p}}}$ for all $z \in \mathbb{D}$.

Proposition 16 *Let $1 < p \leq q < \infty$, $\omega, v \in \mathcal{D}$ and $r \geq r_0$, where $r_0 = r_0(v) > 0$ is that of Theorem 11(i). Then $\text{BMO}(\Delta) \cap \mathcal{H}(\mathbb{D}) = \text{BMO}(\Delta)_{\omega, v, p, q, r} \cap \mathcal{H}(\mathbb{D}) = \mathcal{B}_{d\gamma}$.*

Proof Let first $f \in \mathcal{B}_{d\gamma}$. By Theorem 11(iv) to deduce $f \in \text{BMO}(\Delta)$ it is enough to prove

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|)^{c+1} \gamma(z)^q}{\widehat{v}(z)} \int_{\mathbb{D}} |f(\zeta) - f(z)|^q \frac{(1 - |\zeta|^2)^\sigma}{|1 - z\bar{\zeta}|^{2+c+\sigma}} v(\zeta) dA(\zeta) < \infty \quad (6.1)$$

for some $\sigma > 0$ and

$$c > 2\frac{q}{p}(\beta(\omega) + 1) + \sigma + \max\{2\beta(v), \gamma(v)\}. \quad (6.2)$$

Since $f \in \mathcal{H}(\mathbb{D})$, the function $(f(\zeta) - f(z))(1 - \zeta\bar{z})^{-\frac{2+c+\sigma}{q}}$ is an analytic function in ζ for each $z \in \mathbb{D}$. Therefore Lemma 5 shows that (6.1) is equivalent to

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|)^{c+1} \gamma(z)^q}{\widehat{v}(z)} \int_{\mathbb{D}} |f(\zeta) - f(z)|^q \frac{(1 - |\zeta|^2)^{\sigma-1}}{|1 - z\bar{\zeta}|^{2+c+\sigma}} \widehat{v}(\zeta) dA(\zeta) < \infty. \quad (6.3)$$

Further, Lemma A(ii) yields

$$\begin{aligned} & \frac{(1 - |z|)^{c+1} \gamma(z)^q}{\widehat{v}(z)} \int_{\mathbb{D}} |f(\zeta) - f(z)|^q \frac{(1 - |\zeta|^2)^{\sigma-1}}{|1 - z\bar{\zeta}|^{2+c+\sigma}} \widehat{v}(\zeta) dA(\zeta) \\ & \lesssim (1 - |z|)^{c+1} \gamma(z)^q \int_{\mathbb{D} \setminus D(0, |z|)} |f(\zeta) - f(z)|^q \frac{(1 - |\zeta|^2)^{\sigma-1}}{|1 - z\bar{\zeta}|^{2+c+\sigma}} dA(\zeta) \\ & \quad + (1 - |z|)^{c+1-\beta(v)} \gamma(z)^q \int_{D(0, |z|)} |f(\zeta) - f(z)|^q \frac{(1 - |\zeta|^2)^{\sigma+\beta(v)-1}}{|1 - z\bar{\zeta}|^{2+c+\sigma}} dA(\zeta) \\ & \leq (1 - |z|)^{c+1} \gamma(z)^q \int_{\mathbb{D}} |f(\zeta) - f(z)|^q \frac{(1 - |\zeta|^2)^{\sigma-1}}{|1 - z\bar{\zeta}|^{2+c+\sigma}} dA(\zeta) \\ & \quad + (1 - |z|)^{c+1-\beta(v)} \gamma(z)^q \int_{\mathbb{D}} |f(\zeta) - f(z)|^q \frac{(1 - |\zeta|^2)^{\sigma+\beta(v)-1}}{|1 - z\bar{\zeta}|^{2+c+\sigma}} dA(\zeta) \\ & = I_1(z) + I_2(z), \quad z \in \mathbb{D}. \end{aligned}$$

Fix $\sigma > \max \left\{ 0, 1 - \frac{q}{p}(1 + \alpha(\omega)) + q\beta(v) \right\}$ and c satisfying (6.2). Then

$$c > \max \left\{ \beta(v) - 1, -2 + \beta(v) + \frac{q}{p}(1 + \beta(\omega)) - q\alpha(v) \right\}.$$

Therefore, [11, Lemma 7] together with Lemmas A(ii) and B gives

$$\begin{aligned} I_1(z) &\lesssim (1 - |z|)^{c+1} \gamma(z)^q \int_{\mathbb{D}} |f'(\zeta)|^q \frac{(1 - |\zeta|^2)^{\sigma+q-1}}{|1 - z\bar{\zeta}|^{2+c+\sigma}} dA(\zeta) \\ &\lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q (1 - |z|)^{c+1} \gamma(z)^q \int_{\mathbb{D}} \gamma(\zeta)^{-q} \frac{(1 - |\zeta|^2)^{\sigma-1}}{|1 - z\bar{\zeta}|^{2+c+\sigma}} dA(\zeta) \\ &\asymp \|f\|_{\mathcal{B}_{d\gamma}}^q (1 - |z|)^{c+1} \gamma(z)^q \int_0^1 \frac{\widehat{\omega}(s)^{\frac{q}{p}}}{\widehat{v}(s)} \frac{(1 - s)^{\frac{q}{p} + \sigma - 2}}{(1 - s|z|)^{1+\sigma+c}} ds \\ &\lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q (1 - |z|)^{c+1} \gamma(z)^q \frac{\widehat{\omega}(z)^{\frac{q}{p}} (1 - |z|)^{\alpha(v)}}{(1 - |z|)^{\frac{q}{p}\beta(\omega)} \widehat{v}(z)} \int_0^{|z|} \frac{ds}{(1 - s)^{3+c-\frac{q}{p}-\frac{q}{p}\beta(\omega)+\alpha(v)}} \\ &\quad + \|f\|_{\mathcal{B}_{d\gamma}}^q (1 - |z|)^{-\sigma} \gamma(z)^q \frac{\widehat{\omega}(z)^{\frac{q}{p}} (1 - |z|)^{\beta(v)}}{(1 - |z|)^{\frac{q}{p}\alpha(\omega)} \widehat{v}(z)} \int_{|z|}^1 (1 - s)^{\frac{q}{p} + \sigma - 2 + \frac{q}{p}\alpha(\omega) - \beta(v)} ds \\ &\lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q (1 - |z|)^{-1+\frac{q}{p}} \gamma(z)^q \frac{\widehat{\omega}(z)^{\frac{q}{p}}}{\widehat{v}(z)} \asymp \|f\|_{\mathcal{B}_{d\gamma}}^q < \infty, \quad z \in \mathbb{D}, \end{aligned}$$

and

$$\begin{aligned} I_2(z) &\lesssim (1 - |z|)^{c+1-\beta(v)} \gamma(z)^q \int_{\mathbb{D}} |f'(\zeta)|^q \frac{(1 - |\zeta|^2)^{\sigma+\beta(v)+q-1}}{|1 - z\bar{\zeta}|^{2+c+\sigma}} dA(\zeta) \\ &\lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q (1 - |z|)^{c+1-\beta(v)} \gamma(z)^q \int_{\mathbb{D}} \gamma(\zeta)^{-q} \frac{(1 - |\zeta|^2)^{\beta(v)+\sigma-1}}{|1 - z\bar{\zeta}|^{2+c+\sigma}} dA(\zeta) \\ &\asymp \|f\|_{\mathcal{B}_{d\gamma}}^q (1 - |z|)^{c+1-\beta(v)} \gamma(z)^q \int_0^1 \frac{\widehat{\omega}(s)^{\frac{q}{p}}}{\widehat{v}(s)} \frac{(1 - s)^{\beta(v)+\frac{q}{p}+\sigma-2}}{(1 - s|z|)^{1+\sigma+c}} ds \\ &\lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q (1 - |z|)^{c+1-\beta(v)} \gamma(z)^q \frac{\widehat{\omega}(z)^{\frac{q}{p}} (1 - |z|)^{\alpha(v)}}{(1 - |z|)^{\frac{q}{p}\beta(\omega)} \widehat{v}(z)^q} \int_0^{|z|} \frac{ds}{(1 - s)^{3+c-\beta(v)-\frac{q}{p}-\frac{q}{p}\beta(\omega)+\alpha(v)}} \\ &\quad + \|f\|_{\mathcal{B}_{d\gamma}}^q (1 - |z|)^{-\sigma-\beta(v)} \gamma(z)^q \frac{\widehat{\omega}(z)^{\frac{q}{p}} (1 - |z|)^{\beta(v)}}{(1 - |z|)^{\frac{q}{p}\alpha(\omega)} \widehat{v}(z)} \int_{|z|}^1 (1 - s)^{\beta(v)+\frac{q}{p}+\sigma-2+\frac{q}{p}\alpha(\omega)-\beta(v)} ds \\ &\lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q (1 - |z|)^{-1+\frac{q}{p}} \gamma(z)^q \frac{\widehat{\omega}(z)^{\frac{q}{p}}}{\widehat{v}(z)^q} \asymp \|f\|_{\mathcal{B}_{d\gamma}}^q < \infty, \quad z \in \mathbb{D}. \end{aligned}$$

By combining these estimates we deduce $f \in \text{BMO}(\Delta)$, and thus $\mathcal{B}_{d\gamma} \subset \mathcal{H}(\mathbb{D}) \cap \text{BMO}(\Delta)$.

Assume now that $f \in \mathcal{H}(\mathbb{D}) \cap \text{BMO}(\Delta)$. Then (6.3) holds for some $\sigma > 1$ and c satisfying (6.2). Therefore (3.6) implies

$$\begin{aligned} \infty &> \sup \frac{(1 - |z|)^{c+1} \gamma(z)^q}{\widehat{v}(z)} \int_{\mathbb{D}} |f(\zeta) - f(z)|^q \frac{(1 - |\zeta|^2)^{\sigma-1}}{|1 - z\bar{\zeta}|^{2+c+\sigma}} \widehat{v}(\zeta) dA(\zeta) \\ &\gtrsim \frac{\gamma(z)^q}{(1 - |z|)^2 \widehat{v}(z)} \int_{\Delta(z,r)} |f(\zeta) - f(z)|^q \widehat{v}(\zeta) dA(\zeta) \\ &\asymp \frac{\gamma(z)^q}{|\Delta(z,r)|} \int_{\Delta(z,r)} |f(\zeta) - f(z)|^q dA(\zeta), \quad z \in \mathbb{D}. \end{aligned}$$

By arguing as in [12, 1653–1654] we deduce $\mathcal{H}(\mathbb{D}) \cap \text{BMO}(\Delta) \subset \mathcal{B}_{d\gamma}$. \square

The space $\mathcal{B}_{d\gamma}$ consists of constant functions only if $\limsup_{|z| \rightarrow 1^-} ((1 - |z|)\gamma(|z|))^{-1} = 0$. Moreover, $\mathcal{B}_{d\gamma}$ is a subset of the disc algebra if $((1 - x)\gamma(x))^{-1} \in L^1(0, 1)$, and $\mathcal{B}_{d\gamma}$ coincides with a Bloch-type space if γ is decreasing.

Proof of Theorem 2 Since $f \in A_v^1$, the operator $H_{\bar{f}}^v$ is densely defined. If $H_{\bar{f}}^v : A_{\omega}^p \rightarrow L_v^q$ is bounded, choosing $g \equiv 1$ it follows that $f \in A_v^q$, and therefore $f \in \mathcal{B}_{d\gamma}$ by Theorem 1 and Proposition 16.

Conversely, assume $f \in \mathcal{B}_{d\gamma}$. Since $v \in B_q$ is radial, Proposition 6 implies $v \in \mathcal{D}$. Therefore Lemmas A(ii) and B yield

$$\begin{aligned} \|f\|_{A_v^q}^q &\lesssim \int_{\mathbb{D}} \left(\int_0^{|z|} \left| f' \left(s \frac{z}{|z|} \right) \right| ds \right)^q v(z) dA(z) + |f(0)|^q \\ &\lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q \left(1 + \int_0^1 \left(\int_0^t \frac{ds}{(1-s)\gamma(s)} \right)^q v(t) dt \right) \\ &\lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q \left(1 + \int_0^1 \left(\int_0^t \frac{\widehat{\omega}(s)^{\frac{1}{p}}}{\widehat{v}(s)^{\frac{1}{q}}(1-s)^{1+\frac{1}{q}-\frac{1}{p}}} ds \right)^q v(t) dt \right) \\ &\lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q \left(1 + \int_0^1 \frac{\widehat{\omega}(t)^{\frac{q}{p}}}{\widehat{v}(t)(1-t)^{\frac{q\alpha(\omega)}{p}-\beta(v)}} \left(\int_0^t \frac{ds}{(1-s)^{1+\frac{1+\beta(v)}{q}-\frac{1+\alpha(\omega)}{p}}} \right)^q v(t) dt \right) \end{aligned}$$

for all $f \in \mathcal{H}(\mathbb{D})$. If $\frac{1+\beta(v)}{q} - \frac{1+\alpha(\omega)}{p} > 0$, Lemma 3(ii) gives

$$\begin{aligned} \|f\|_{A_v^q}^q &\lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q \left(1 + \int_0^1 \frac{\widehat{\omega}(t)^{\frac{q}{p}}(1-t)^{\frac{q}{p}-1}}{\widehat{v}(t)} v(t) dt \right) \\ &\lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q \left(1 + \widehat{\omega}(0)^{\frac{q}{p}-1} \int_0^1 \frac{\widehat{\omega}(t)v(t)}{\widehat{v}(t)} dt \right) \lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q. \end{aligned}$$

If $\frac{1+\beta(v)}{q} - \frac{1+\alpha(\omega)}{p} = 0$, then Lemmas B and 3(ii) yield

$$\begin{aligned} \|f\|_{A_v^q}^q &\lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q \left(1 + \int_0^1 \frac{\widehat{\omega}(t)^{\frac{q}{p}}(1-t)^{\frac{q}{p}-1}}{\widehat{v}(t)} \log \frac{e}{1-t} v(t) dt \right) \\ &\lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q \widehat{\omega}(0)^{\frac{q}{p}} \int_0^1 \frac{(1-t)^{\alpha(\omega)\frac{q}{p}+\frac{q}{p}-1}}{\widehat{v}(t)} \log \frac{e}{1-t} v(t) dt \\ &\lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q \int_0^1 \frac{(1-t)^{\alpha(\omega)\frac{q}{2p}+\frac{q}{p}-1}}{\widehat{v}(t)} v(t) dt \lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q. \end{aligned}$$

Finally, if $\frac{1+\beta(v)}{q} - \frac{1+\alpha(\omega)}{p} < 0$, then Lemma 3(ii) gives

$$\|f\|_{A_v^q}^q \lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q \left(1 + \int_0^1 \frac{\widehat{\omega}(t)^{\frac{q}{p}}(1-t)^{\frac{q}{p}-1}}{\widehat{v}(t)} v(t) dt \right) \lesssim \|f\|_{\mathcal{B}_{d\gamma}}^q.$$

Therefore $f \in A_v^q$, and thus $\mathcal{B}_{d\gamma} \subset A_v^q$. This together with Theorem 1 and Proposition 16 finishes the proof. \square

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